# Math 121A Spring 2009
## Homework#9 Fourier transform, Series and Parseval's theorem.

1. Consider the following sets $L^2(T)$ of all functions from $[-\pi, \pi]$ to $\mathbb{C}$ that satisfy $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx < \infty$ and the set $l^2(\mathbb{Z})$ of all sequences $(a_n)_{n\in\mathbb{Z}}$ of complex numbers such that $\sum_{n\in\mathbb{Z}} |a_n|^2 < \infty$. Show that $L^2(T)$ and $l^2(\mathbb{Z})$ are vector spaces over $\mathbb{C}$ (Clue: Triangle inequality). What is the dimension of these vector spaces?

2. For a function $f \in L^2(T)$ we define its *Fourier transform* $\widehat{f}$ to be the sequence $(\widehat{f}(n))$ with $\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$. We define the Fourier series of $f$ to be the series $\sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{inx}$. Compute the Fourier series of the following functions

   (a) $f(x) = x^2$.

   (b) $f(x) = |x|$.

   (c) $f(x) = 0$ for $-\pi \le x \le 0$ and $f(x) = \sin(x)$ for $0 \le x \le \pi$.

3. Consider the inner products $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx$ on $L^2(T)$ and $[(a_n), (b_n)] = \sum_{n\in\mathbb{Z}} a_n\overline{b_n}$ on $l^2(\mathbb{Z})$. We have the following Theorem

**Theorem** (Parseval). The Fourier transform$\widehat{\ }$: $L^2(T) \to l^2(\mathbb{Z})$ is an isomorphism of inner product spaces. i.e., Fourier transform is a linear operator, it is $1-1$ and onto and for every $f, g \in L^2(T)$ we have $\langle f, g\rangle = \left[(\widehat{f}(n)), (\widehat{g}(n))\right]$.

- Show that in particular for every $f \in L^2(T)$ we have $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2$ (*)

- Use the identity (*) and the function $f : [-\pi, \pi] \to \mathbb{C}$, $f(x) = x$ to verify the identity $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$.

- Find what identities follows from (a) above when applied to the functions appearing in 2.($a$), 2.($b$) and 2.($c$) above.

- **Remarks**
  - You are very much encouraged to work with other students. However, submit your work alone.
  - I will be happy to help you with the homeworks. Please visit me if you want to work with me.

**Good luck!**