Math 121A Spring 2007

Homework#7 How to think on analytic function? Taylor series.

1. (Topology) A subset $U \subset \mathbb{C}$ is called *open* if for every $z_0 \in \mathbb{C}$ there exists a ball $B = B(z_0, r) = \{z \in \mathbb{C}; |z - z_0| < r\}$ with $z_0 \in B \subsetneq U$. Draw the following sets and decide if they are open or not

   (a) The ball $B(0, 1)$.

   (b) The complex plane.

   (c) The close ball $\overline{B}(0, 1) = \{z \in \mathbb{C}; |z| \leq 1\}$.

2. (How to think on analytic function?) Recall the theorem: Let f be an analytic function in an open subset $U \subset \mathbb{C}$ and let $z_0 \in \mathbb{C}$. Consider a ball B such that $z_0 \in B \subset U$. Then in B the function f can be expanded uniquely as an absolutely convergent series called Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n. \tag{1}$$

   For the following functions find the appropriate Taylor series and write its ball of convergence.

   (a) $f(z) = \frac{1}{1+z^3}$ around $z_0 = 0$.

   (b) $f(z) = \frac{1}{z}$ around $z_0 = 1$.

   (c) $f(z) = \frac{1}{z(z-2)}$ around $z_0 = 1$.

3. Suppose that a complex function f can be written as a Taylor series (1) around some point z_0. Show that $a_n = \frac{f^{(n)}}{n!}$. Explain!.

4. Use the Taylor expansion and its uniqueness to compute the derivatives $f^{(n)}(0)$, $n \geq 0$, of $f(z) = \frac{1}{1+z^2}$.

5. Find ∞ many "interesting" functions $u : \mathbb{R}^2 \to \mathbb{R}$ that solve the Laplace equation

$$\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} = 0.$$

- **Remarks**
  - You are very much encouraged to work with other students. However, submit your work alone.
  - I will be happy to help you with the homeworks. Please visit me if you want to work with me.

Good luck in our Mid-Term test!