Math 121A Spring 2009

Homework#5 Derivative, C-R equations.

1. Compute from the definition
$$f'(z) = \lim_{w \to 0} \frac{f(z+w) - f(z)}{w},$$
that $(z^n)' = nz^{n-1}$.

2. Use linearity, the quotient rule, the chain rule and the Leibniz rule to calculate the derivatives of
   (a) $e^{z^3 \cos(z)}$.
   (b) $\frac{\cos^2(z)+\sin^2(z)}{z}$.
   (c) $\ln(z)$.
   (d) General power series
   $$s(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$
   in the disc of convergence $U = \{z \in \mathbb{C};\ |z - z_0| < R\}$. Explain where you use absolutely convergence of the series.
   (e) Compute the radius R and the disc of convergence U of the power series
   $$s(z) = \sum_{n=1}^{\infty} \frac{(z-1)^n}{n^2}.$$
   Compute $s'(z)$ inside U.

3. Formulate the theorem on Cauchy-Riemann equations. Show that the converse of the theorem is not true: Consider the function $f : \mathbb{C} \to \mathbb{C}$ given by $f(x+iy) = 1$ if $x \neq 0$, $y \neq 0$ and $f(x+iy) = 0$ otherwise. Show that f satisfies the C-R at $(0,0)$ but is not analytic there (it is even not continuous at $(0,0)$).

4. Write our definition of the function $f(z) = \sqrt{z}$. Explain why f is not continuous at all points $(x, 0)$ with $x \geq 0$. Verify C-R equations for f at all other points, i.e., in $U = \mathbb{C} - \{(x,0);\ x \geq 0\}$.

5. Two real functions $u(x,y)$ and $v(x,y)$ defined on a subset $U \subset \mathbb{R}^2$ are called C-R pair if they satisfy the C-R equations in U. Find ∞ many C-R pairs on $\mathbb{R}^2$ (Clue: read C-R theorem from the end to the beginning and consider polynomials).

- **Remarks**
  - You are very much encouraged to work with other students. However, submit your work alone.
  - I will be happy to help you with the homeworks. Please visit me if you want to work with me.

Good Luck!