Math 113 Spring 2007
HW#6: Groups and cyclic groups

1. Compute the following subgroups of $\mathbb{Z}$:
   (a) $H = 3\mathbb{Z} + 4\mathbb{Z}$.
   (b) $H = 3\mathbb{Z} + 6\mathbb{Z}$.
   (c) $H = m\mathbb{Z} + n\mathbb{Z}$ where m, n have no common divisor other then ± 1.

2. Show that any group with exactly 3 element is isomorphic to the group $\mathbb{Z}_3$.

3. Determine the elements of the cyclic subgroup generated by $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$.

4. Let G be a group and $g, h \in G$. The subgroup generated by g and h which we denote by $\langle g, h \rangle$ is defined to be the smallest subgroup of G which contains g and h. Show that if $gh = hg$ then $\langle g, h \rangle$ is Abelian.

5. We say that a subgroup $H \subset G$ is a proper subgroup if $1 \subsetneq H \subsetneq G$. Describe all groups G with no proper subgroup (Clue: Show that G must be cyclic group of prime order, i.e., $G \simeq \mathbb{Z}_p$ where p is a prime number).

6. Show that every subgroup of a cyclic group is cyclic (Clue: if $1 \subsetneq H \subsetneq \langle x \rangle$ then there exist a minimal $k > 1$ such that $x^k \in H$. Show $\langle x^k \rangle = H$).

7. Consider the Klein four group

$$K = \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \right\}.$$

   (a) Show that K is not cyclic (In fact, this is the smallest group which is not cyclic).
   (b) Show that K act on the set $X = \{(1,0), (0,1)(-1,0)(0,-1)\} \subset \mathbb{R}^2$.
   (c) Show that $K \simeq \mathbb{Z}_2 \times Z_2$ using an explicit isomorphism.

- **Remarks**

- You are very much encouraged to work with other students. However, submit your work alone.
  - I will be happy to help you with the homeworks. Please visit me if you want to work with me.

Good Luck!