

**Math 113 Spring 2007**  
**HW#5: Subgroups.**

1. For each of the following decide if it is a subgroup.
  - (a) The subset  $H \subset S_5 = \text{Aut}(\{1, \dots, 5\})$  with  $H = \{\sigma \in S_5; \sigma(1) = 1\}$ .
  - (b) The subset  $\mathbb{R}^\times = \mathbb{R} - \{0\}$  in  $\mathbb{C}^\times = \mathbb{C} - \{0\}$  with multiplication.
  - (c) The subset of positive real numbers in  $\mathbb{R}^\times$  with multiplication.
  - (d) The subset  $\mu_n = \{z \in \mathbb{C}^\times; z^n = 1\}$  in  $\mathbb{C}^\times$ .
  - (e) The subset  $T \subset GL_2(\mathbb{R})$  with  $T = \left\{ \begin{pmatrix} a & \\ & b \end{pmatrix}; ab \neq 0 \right\}$ .
2. Let  $G$  be a group and  $X$  a set. An action of  $G$  on  $X$  is an assignment for every  $g \in G$  and every  $x \in X$  an element  $g \cdot x \in X$  such that  $(gg') \cdot x = g \cdot (g' \cdot x)$  and  $1_G \cdot x = x$ . Show that
  - (a) If  $G$  act on  $X$  then for every  $g \in G$  the induced map  $\alpha_g : X \rightarrow X$  given by  $\alpha_g(x) = g \cdot x$  is a bijection.
  - (b) If  $G$  act on  $X$  then for every subset  $Y \subset X$  the *stabilizer*  $H = \text{Stab}_G(Y) = \{g \in G; g \cdot Y = Y\}$  (i.e., the set of all  $g \in G$  with  $\alpha_g$  restricted to  $Y$  is a bijection of  $Y$ ) is a subgroup of  $G$ .
  - (c) Use the notion of stabilizer to show that the set  $T = \left\{ \begin{pmatrix} a & \\ & b \end{pmatrix}; ab \neq 0 \right\} \subset GL_2(\mathbb{R})$  is a subgroup (Clue:  $G = GL_2(\mathbb{R})$  with its diagonal action on  $\mathbb{R}^2 \times \mathbb{R}^2$  via  $g \cdot ((x, y), (x', y')) = (g \cdot (x, y), g \cdot (x', y'))$  and  $Y = \{((x, 0), (0, y')); x, y' \in \mathbb{R}\}$ ).
3. Let  $G$  be a group and consider the set  $Z(G) = \{h \in G; hg = gh \text{ for every } g \in G\}$  called the *center* of  $G$ . Show that  $Z(G)$  is a subgroup of  $G$ .
4. Compute the center  $Z(G)$  for  $G = S_3$  and for  $G = SL_2(\mathbb{R}) = \{A \in GL_2(\mathbb{R}); \det(A) = 1\}$ .
5. Is the set  $\left\{ \begin{pmatrix} a & \\ & 0 \end{pmatrix}; a \neq 0 \right\}$  a group with matrix multiplication? Is it a subgroup of  $GL_2(\mathbb{R})$ ?

• **Remarks**

- You are very much encouraged to work with other students. However, submit your work alone.
- I will be happy to help you with the homeworks. Please visit me if you want to work with me.

**Good Luck!**