Math 104, Spring 09
Homework#8: Series and continues functions

1. Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radios of convergence $R = 2$. Assume in addition that $a_1 \geq a_2 \geq ..... \to 0$. Show that $A(z)$ converge for every z with $|z| = 2$ except possibly for $z = 2$.

2. Consider the set of all absolutely convergent series $A = \{\sum_{n=0}^{\infty} a_n$; such that $\sum_{n=0}^{\infty} |a_n| < \infty\}$. Show that A is an algebra, i.e., it is closed under the following operations:

   (a) Addition.

   (b) Multiplication by a scalar.

   (c) Multiplication. Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be two elements of A. Define a new series $\sum_{n=0}^{\infty} c_n$ with $c_n = \sum_{k=0}^{n} a_n b_{n-k}$. You should prove that A is also a member of A (You can use the proof on page 74 of the book. Please try to explain in your answer what is the main point in the proof).

3. Write the graph of $f(x) = \frac{x^2-2}{x-2}$.

4. Recall the definition of the limit of a function $f : X \to Y$ where (X, d_X) and (Y, d_Y) are two metric spaces.

5. (Alternative definition of a limit in terms of sequences). Let $f : X \to Y$ be a function between two metric spaces. Show that $\lim_{x \to x'} f(x') = y'$ if and only if for every sequence $(x_n) \subset X$ with $x_n \to x'$ we have $f(x_n) \to y'$.

6. Let X be a metric space. Use 5. to show that the set $A = \{f : X \to \mathbb{C}$; such that f is continuos$\}$ is an algebra with the addition, multiplication by a scalar, and multiplication are defined by $(f + g)(x) = f(x) + g(x)$, $(\alpha f)(x) = \alpha f(x)$ for every $\alpha \in \mathbb{C}$ and finally $(f \cdot g)(x) = f(x)g(x)$ for every $f, g \in A$.

Good luck!!