Math 104, Spring 09

Homework#7: Series and power series

1. Proof that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverge.

2. We defined the number $e = \sum_{n=1}^{\infty} \frac{1}{n}$. Show (you can use the book) that $\left(1 + \frac{1}{n}\right)^n \to e$ as $n \to \infty$.

3. Proof the following theorem: Let $\sum_{n=0}^{\infty} a_n z^n$ be a formal power series with $a_n \in \mathbb{C}$. Denote by $\alpha = \limsup |\frac{a_{n+1}}{a_n}|$ and by $R = \frac{1}{\alpha}$ (with the conventions $\alpha = 0$ then $R = \infty$ and $\alpha = \infty$ then $R = 0$). Then $\sum_{n=0}^{\infty} a_n z^n$ converge if $|Z| < R$ and diverge for $|Z| < R$.

4. Consider the formal power series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$. Show that for every complex number $z \in \mathbb{C}$ this power series absolutely converge. We will denote that number by e^z. Show that

   (a) For every $z, w \in \mathbb{C}$ we have the identity $e^{z+w} = e^z \cdot e^w$. This is called De-Moivre's theorem and believe it or not it is deep. Pay attention in your proof where you use the fact that the series e^z is absolutely convergence. The nice thing about absolutely convergent series is that you can organize its terms as you wish and still the series will converge to the same number.

   (b) Show that for a real number θ one has $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. Probably you want to recall the identity $\cos(\theta) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$ and $\sin(\theta) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$. This is known as Euler's theorem.

   (c) Find using D-M's and Euler's theorems the formulas for $\sin(\theta+\varphi) =?$ and $\cos(\theta+\varphi) =?$.

5. Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converge only conditionally. What theorem you use?

6. Show that the power series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converge for every $|z| \leq 1$. What happens if $|Z| > 1$?

Good luck!!