A note on the L^p bounds of the Hilbert transform along $C^{1+\eta}$ vector fields

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Abstract

In this note we will apply Bateman and Thiele's square function estimate in [3] to answer one question posed by Lacey and Li in [5] on the L^p boundedness of the Hilbert transform along $C^{1+\eta}$ vector fields for any $\eta > 0$, conditioning on the boundedness of the Lipschitz-Kakeya maximal operator. The main observation here is that an L^p variant of the Cotlar-Stein Lemma in [4] can be applied.

1 Introduction

For a given unit vector field v on the plane, define the Hilbert transform along this vector field with cut-off ϵ_0 by

$$(H_{v,\epsilon_0} f)(x) := \int_{-\epsilon_0}^{\epsilon_0} f(x - tv(x)) \frac{1}{t} dt. \tag{1.1}$$

In [5], the following conjecture is stated:

Conjecture 1.1. *There exists $\kappa > 0$ such that for any Lipschitz vector field v with Lipschitz constant $\|v\|_{Lip}$, the associated Hilbert transform H_{v,ϵ_0} with $\epsilon_0 := \kappa/\|v\|_{Lip}$ is of weak type (2,2), i.e. we have that*

$$\sup_{\lambda>0} \lambda |\{x : |(H_{v,\epsilon_0} f)(x)| > \lambda\}|^{1/2} \lesssim \|f\|_2. \tag{1.2}$$

For $C^{1+\eta}$ vector fields with $\eta > 0$, Lacey and Li in their prominent work [5] have reduced the L^2 boundedness of the operator H_{v,ϵ_0} to the boundedness of one new operator they introduced, the so called Lipschitz-Kakeya maximal operator. In order to define this maximal operator, we first need to recall some definitions in [5].

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A *rectangle* is determined as follows. Fix a choice of unit vectors in the plane $(e, e^{\perp})$, with $e^{\perp}$ being the vector e rotated by $\pi/2$. Using these vectors as coordinate axis, a rectangle is a product of two intervals $R = I \times J$. We will use $L(R) = |I|, W(R) = |J|$ to denote the length and width of R respectively.

The *interval of uncertainty* of R is the subarc $EX(R)$ of the unit circle in the plane, centered at e, and of length $W(R)/L(R)$.

For any Lipschitz unit vector field v, due to the cut-off of the Hilbert transform, we only consider rectangles R with

$$L(R) \leq (100\|v\|_{Lip})^{-1}. \tag{1.3}$$

For such a rectangle, we set $V(R) = R \cap v^{-1}(EX(R))$. Then the Lipschitz-Kakeya maximal operator is defined by

$$M_{v,\delta,w} f(x) := \sup_{|V(R)| \geq \delta |R|, W(R) = w} \frac{\chi_R(x)}{|R|} \int_R |f(y)| dy, \tag{1.4}$$

for $0 < \delta \leq 1$ and $0 < \omega \ll 1$.

Conditioning on the boundedness of the Lipschitz-Kakeya maximal operator, Lacey and Li proved the following result.

Theorem 1.2. *([5]) Assume that for some $p \in (1,2)$ there exists $N > 0$ such that*

$$\|M_{v,\delta,w}\|_{p \to p} \lesssim \delta^{-N}, \tag{1.5}$$

with constant independent of $0 < \delta \leq 1$ and $0 < w < (100\|v\|_{Lip})^{-1}$. Then there exists $K > 0$ such that for any $C^{1+\eta}$ unit vector field v with $\eta > 0$, the operator H_{v,ϵ_0} is bounded in L^2 with ϵ_0 given by $\frac{K}{\|v\|_{C^{1+\eta}}}$.

In the proof of the above theorem, Lacey and Li carried out a Schur's test to explore the almost orthogonality between different frequency annuli, hence the argument there only works in the case of L^2. Indeed, they raised the L^p boundedness as the following conjecture.

Conjecture 1.3. *([5]) Under the same assumptions as in the Theorem 1.2, H_{v,ϵ_0} is also bounded in L^p for all $2 < p < \infty$.*

Here we will give an affirmative answer to the above conjecture, under a slightly stronger assumption, which is the boundedness of a variant of the Lipschitz-Kakeya maximal operator used by Bateman in the case of one variable vector fields in [1] and [2]. To introduce this variant, we need one more definition.

Definition 1.4. *Given two rectangles R_1 and R_2 in $\mathbf{R}^2$, we write $R_1 \leq R_2$ whenever $R_1 \subset CR_2$ and $EX(R_2) \subset EX(R_1)$, where C is some properly chosed large constant, and CR_2 is the rectangle with the same center as R_2 but dilated by the factor C.*

Similar to [5], we formulate the boundedness of the variant of the Lipschitz-Kakeya maximal operator in the following conjecture.

Conjecture 1.5. *For any Lipschitz unit vector field v, suppose $\mathcal{R}_0$ is a collection of pairwise incomparable (under "$\leq$") rectangles of uniform width and of length less than $(100\|v\|_{Lip})^{-1}$, such that for each $R \in \mathcal{R}_0$, we have*

$$\frac{|V(R)|}{R} \geq \delta, \tag{1.6}$$

and

$$\frac{1}{|R|}\int_R \chi_F \geq \lambda. \tag{1.7}$$

Then for each $p > 1$,

$$\sum_{R\in\mathcal{R}_0} |R| \lesssim \frac{|F|}{\delta\lambda^p}. \tag{1.8}$$

Remark 1.6. *Bateman verified the truth of this conjecture in [1] for the one variable vector fields. On that basis, Bateman [2], Bateman and Thiele [3] proved that the Hilbert transform (without cut-off) along the one variable vector fields is bounded in L^p for all $p > 3/2$.*

Remark 1.7. *If the above conjecture holds true, then the same argument in [1] will imply that Lacey and Li's Lipschitz-Kakeya maximal operator is also bounded in L^p for all $p > 1$.*

Now we are ready to state our main result.

Theorem 1.8 (Main Theorem). *Assume that the Conjecture 1.5 holds true, then for any $C^{1+\eta}$ vector field v with $\eta > 0$, the associated Hilbert transform H_{v,ϵ_0}, with ϵ_0 given by $\frac{K}{\|v\|_{C^{1+\eta}}}$, is bounded in L^p for all $p > 3/2$.*

Remark 1.9. *The restriction that $p > 3/2$ comes from Bateman and Thiele [3] as we are using their result as a black box.*

The key observation in the proof of the Main Theorem is, that together with the square function estimate by Bateman and Thiele in [3], an L^p variant of the Cotlar-Stein Lemma, which is formulated in [4], can be applied.

In Section 2 we will state this almost orthogonaity lemma and show how it implies the L^p boundedness of H_{v,ϵ_0}. In Section 3 and 4 we will verify the two conditions needed to apply the almost orthogonality lemma separately.

Throughout this paper, we will use the same notations as in [5].

2 One almost orthogonality lemma

In this section, we will show the boundedness of H_{v,ϵ_0} as a corollary of an almost orthogonality lemma.

By scaling, we can assume that $\|v\|_{C^{1+\eta}} = 1$. Also later for the sake of convenience we will just write H_v instead of H_{v,ϵ_0}. By losing a factor of eight, we can restrict the action of the operator to functions with frequency supported only in the cone Γ which forms an angle less than $\frac{\pi}{8}$ with the positive half of the vertical axis. Hence later whenever we write the operator P_k, we mean $P_k \circ \Pi_\Gamma$, where P_k is the (homogeneous) Littlewood-Paley projection operator and Π_Γ is the frequency projection operator $\Pi_\Gamma := \mathcal{F}^{-1}\chi_\Gamma\mathcal{F}$.

By the normalization $\|v\|_{C^{1+\eta}} = 1$, in the definition of the operator H_v, we restrict the Hilbert kernel dt/t to the region $\{t : |t| \lesssim 1\}$, which is the high frequency part of the kernel. This means the low frequency part of the function will play no role as it cancels out with the high frequency kernel, hence the operator H_v is essentially equal to

$$\sum_{k\in\mathbf{Z},k\geq 0} H_v P_k, \tag{2.1}$$

which can be further written as the model sum

$$\sum_{ann\geq 1}\ \sum_{s\in\mathcal{AT}(ann),scl(s)\geq 1} \langle f, \varphi_s\rangle \phi_s, \tag{2.2}$$

by averaging over translations, dilations and rotations. For the details and the notations we refer to [5].

Now comes the almost orthogonality lemma. In the following, for $k \in \mathbf{N}$, we use P_k^v to denote the non-homogeneous Littlewood-Paley projection operator along the vertical direction.

Lemma 2.1. *([4]) Assume that there exists constant $\gamma > 0$ such that*

$$\|P_j^v H_v P_k\|_{L^2\to L^2} \lesssim 2^{-\gamma|j-k|}, \forall k, j \in \mathbf{N}, \tag{2.3}$$

and for some $p_0 \neq 2$

$$\|(\sum_{k\geq 0} |H_v P_k g_k|^2)^{1/2}\|_{p_0} \lesssim \|(\sum_{k\geq 0} |g_k|^2)^{1/2}\|_{p_0}, \tag{2.4}$$

then $\sum_{k\geq 0} H_v P_k$ is bounded in L^p for p between 2 and p_0.

Proof of Lemma 2.1: the proof is almost the same with the one in [4], except that we are using the Littlewood-Paley projection operator along the vertical direction P_k^v. We include the proof here for completeness.

By the Littlewood-Paley theory along the vertical direction and the triangle inequality, we have

$$\begin{aligned}\|\sum_{k\geq 0} H_v P_k f\|_p &\lesssim \|(\sum_{j\geq 0} |P_j^v[\sum_{k\geq 0} H_v P_k f]|^2)^{1/2}\|_p \\ &\lesssim \sum_{j\in\mathbf{Z}} \|(\sum_{k\geq \max\{-j,0\}} |P_{k+j}^v H_v P_k f|^2)^{1/2}\|_p.\end{aligned}$$

For $p = 2$,

$$\begin{aligned}&\|(\sum_{k\geq \max\{0,-j\}} |P_{k+j}^v H_v P_k f|^2)^{1/2}\|_2^2 \\ &= \int_{\mathbf{R}^2} \sum_{k\geq \max\{0,-j\}} |P_{k+j}^v H_v P_k f|^2 \\ &\lesssim \int_{\mathbf{R}^2} \sum_{k\geq \max\{0,-j\}} 2^{-2\gamma|j|} |P_k f|^2 \lesssim 2^{-2\gamma|j|} \|f\|_2^2,\end{aligned}$$

where in the second last step we applied the almost orthogonality condition (2.3). For $p = p_0$,

$$\begin{aligned}&\|(\sum_{k\geq \max\{0,-j\}} |P_{k+j}^v H_v P_k(f)|^2)^{1/2}\|_{p_0} \\ &\lesssim \|(\sum_{k\geq \max\{0,-j\}} |H_v P_k(f)|^2)^{1/2}\|_{p_0} \lesssim \|f\|_{p_0},\end{aligned}$$

where in the last step we used the square function estimate (2.4). Interpolation of the above two estimates gives

$$\|(\sum_{k\geq \max\{0,-j\}} |P_{k+j}^v H_v P_k(f)|^2)^{1/2}\|_p \lesssim 2^{-\gamma_p|j|} \|f\|_p, \tag{2.5}$$

and summing over j gives us the desired bound for $\sum_{k\geq 0} H_v P_k$.□

Hence to prove the Main Theorem, we just need to verify the two conditions needed to apply Lemma 2.1, which will be done separately in the following two sections.

3 Square function estimate (2.4)

Lemma 3.1. *Under the same assumptions as in the Main Theorem, we have that the following square function estimate holds true for all $p_0 > 3/2$:*

$$\|(\sum_{k\geq 0} |H_v P_k(g_k)|^2)^{1/2}\|_{p_0} \lesssim \|(\sum_{k\geq 0} |g_k|^2)^{1/2}\|_{p_0}. \tag{3.1}$$

Proof of Lemma 3.1: the proof is due to Bateman [2], Bateman and Thiele [3].We just need to replace the key lemmas used in [2] and [3] by the corresponding versions for Lipschitz vector fields.

First, in Bateman [2], the only place that the one variable vector field plays a special role is in the verification of the Conjecture 1.5. Here as we are still proving a conditional result, we just need to replace the Lemma 20 in [2] by the assumption that the Conjecture 1.5 holds true for general Lipschitz vector fields, and leave the rest of the argument unchanged, then all the estimates stated in [2] will remain.

Second, in Bateman and Thiele [3], we just need to replace Lemma 8 by the "weak L^2 estimate for the Lipschitz-Kakeya maximal function 1.10" in Page 3 of [5], as the rest of the argument also works for general Lipschitz vector fields.

Hence we have proved the square function estimate (3.1). □

4 Almost orthogonality condition (2.3)

Lemma 4.1. *For any $C^{1+\eta}$ unit vector field v with $\|v\|_{C^{1+\eta}} = 1$, there exists constant $\gamma > 0$ depending only on η s.t.*

$$\|P_j^v H_v P_k\|_{L^2 \to L^2} \lesssim 2^{-\gamma|j-k|}, \forall j, k \in \mathbf{N}. \tag{4.1}$$

Proof of Lemma 4.1: The above estimate is essentially Lemma 5.4 in Lacey and Li [5], which is again a corollary of the Fourier localization lemma 5.56 in the same paper. Indeed we can show more than claimed, which is

$$\|P_j^v H_v P_k\|_{L^2 \to L^2} \lesssim 2^{-\gamma \max\{j,k\}}, \forall j \neq k \in \mathbf{N}. \tag{4.2}$$

If we write the last expression in the model sum, what we need to prove turns to

$$\| \sum_{s \in \mathcal{AT}(2^k), 1 \leq scl(s) \leq 2^k} \langle f, \varphi_s \rangle P_j^v(\phi_s)\|_2 \lesssim 2^{-\gamma \max\{k,j\}} \|f\|_2. \tag{4.3}$$

Notice that by the restriction $1 \leq scl(s) \leq 2^k$, we have at most k different scales in the summation in (4.3), hence by the triangle inequality, it suffices to prove

$$\| \sum_{s \in \mathcal{AT}(2^k), scl(s) = scl} \langle f, \varphi_s \rangle P_j^v(\phi_s)\|_2 \lesssim 2^{-\gamma \max\{k,j\}} \|f\|_2, \tag{4.4}$$

for some $\gamma > 0$, with constant independent of $scl \in [1, 2^k]$ and scl being a dyadic number.

Case $j < k$: the idea is that ϕ_s should be considered as a function with frequency supported essentially on the k-th annulus, then the action of P_j^v

on ϕ_s will give the exponential decay by Lacey and Li's Fourier localization lemma.

To be precise, let ρ be a smooth function on $\mathbf{R}$, with

$$\chi_{(\frac{1}{4},4)} \leq \hat{\rho} \leq \chi_{(\frac{1}{2},2)}, \tag{4.5}$$

and set $\rho_k(t) = 2^k \rho(2^k t)$. We subtract

$$\int \phi_s(x - te^{\perp})\rho_k(t)dt \tag{4.6}$$

from the term ϕ_s, then by the fact that

$$P_j^v[\int \phi_s(x - te^{\perp})\rho_k(t)dt] = 0, \tag{4.7}$$

we know

$$P_j^v(\phi_s) = P_j^v[\phi_s - \int \phi_s(x - te^{\perp})\rho_k(t)dt], \tag{4.8}$$

which turns the model sum to

$$\sum_{s \in \mathcal{AT}(2^k), scl(s) = scl} \langle f, \varphi_s \rangle P_j^v[\phi_s - \int \phi_s(x - te^{\perp})\rho_k(t)dt]. \tag{4.9}$$

Remark 4.2. *In* (4.7) *we can see the point of restricting the frequency support of the function to the cone which forms an angle less than* $\pi/8$ *with the positive half of the vertical axis, and of using the non-homogeneous Littlewood-Paley projection operator* P_v^j *along the vertical axis.*

In order to apply the Fourier localization lemma in [5], we write the term we need to bound in the above expression in a slightly different form, which is

$$\begin{aligned} &\sum_{s \in \mathcal{AT}(2^k), scl(s) = scl} \langle f, \varphi_s \rangle (\phi_s - \int \phi_s(x - te^{\perp})\rho_k(t)dt) \\ = &\sum_{s \in \mathcal{AT}(2^k), scl(s) = scl} \langle f, \varphi_s \rangle Mod_{c(\omega_s)}(Mod_{-c(w_s)}\phi_s - Mod_{-c(w_s)} \int_{\mathbf{R}} \phi_s(x - te^{\perp})\rho_k(t)dt) \\ = &\sum_{s \in \mathcal{AT}(2^k), scl(s) = scl} \langle f, \varphi_s \rangle Mod_{c(\omega_s)}(f_s - \int_{\mathbf{R}} f_s(x - te^{\perp}) e^{-it\langle e^{\perp}, c(w_s)\rangle} \rho_k(t)dt), \end{aligned}$$

where $f_s := Mod_{-c(w_s)}\phi_s$, following Lacey and Li's notation. Then if we define

$$\zeta(t) := e^{-it\langle e^{\perp}, c(w_s)\rangle} \rho_k(t), \tag{4.10}$$

the Fourier transform of the function ζ will satisfy the condition of Fourier localization lemma in [5], which gives the following pointwise estimate

Lemma 4.3. *([5]) Under the above notations, and for $\epsilon < \frac{\eta}{20}$, we have*

$$|f_s(x) - \int_{\mathbf{R}} f_s(x - te^{\perp})\zeta(t)dt|$$
$$\lesssim (scl \cdot 2^{\eta k})^{-1+\epsilon}\chi^{(2)}_{R_s}(x)\chi_{\tilde{\omega}_s}(v(x)) + (2^k scl)^{-10}\chi^{(2)}_{R_s}(x) + |R_s|^{-1/2}\chi_{F_s}(x),$$

where $\chi^{(2)}_{R_s}$ is the L^2 normalized function associated to R_s, $\tilde{\omega}_s$ is a sub arc of the unit circle, with $\tilde{\omega}_s = \lambda\omega_s$, and $1 < \lambda < 2^{\epsilon k}$. Moreover the set $F_s \subset v^{-1}(\tilde{\omega}_s)$ satisfies the estimate

$$|F_s| \lesssim 2^{-(1+\eta-\epsilon)k}|R_s|. \tag{4.11}$$

Continue the calculation before Lemma 4.3: for the three terms appearing in Lemma 4.3, the middle term $(2^k scl)^{-10}\chi^{(2)}_{R_s}(x)$ will never cause a problem because of the high power of decay in k, hence we will leave out the proof for this term.

For the first term, we have

$$\| \sum_{s\in\mathcal{AT}(2^k), scl(s)=scl} \langle f, \varphi_s\rangle (scl \cdot 2^{\eta k})^{-1+\epsilon}\chi^{(2)}_{R_s}(x)\chi_{\tilde{\omega}_s}(v(x))\|_2^2$$
$$\lesssim \sum_{s\in\mathcal{AT}(2^k), scl(s)=scl} |\langle f, \varphi_s\rangle|^2 2^{-2\eta(1-\epsilon)k}2^{\epsilon k},$$

where we have the factor $2^{\epsilon k}$ due to the fact that ω_s is enlarged $\lambda \leq 2^{\epsilon k}$ times, which means every tile will be counted for at most $2^{\epsilon k}$ times. Then by orthogonality, we have

$$\sum_{s\in\mathcal{AT}(2^k), scl(s)=scl} |\langle f, \varphi_s\rangle|^2 2^{-2\eta(1-\epsilon)k}2^{\epsilon k} \lesssim 2^{-2\gamma k}\|f\|_2^2, \tag{4.12}$$

with $\gamma := \eta(1-\epsilon) - \epsilon/2 > 0$.

For the third term, similar idea gives that

$$\| \sum_{s\in\mathcal{AT}(2^k), scl(s)=scl} \langle f, \varphi_s\rangle |R_s|^{-1/2}\chi_{F_s}(x)\|_2^2$$
$$\lesssim \sum_{s\in\mathcal{AT}(2^k), scl(s)=scl} |\langle f, \varphi_s\rangle|^2 |R_s|^{-1}|F_s|2^{\epsilon k} \lesssim 2^{-(1+\eta)k}\|f\|_2^2 \lesssim 2^{-2\gamma k}\|f\|_2^2.$$

So far we have finished the proof for the case $j < k$.

Case $j > k$: in this case we need both the frequency seperation of P^v_j with P_k, and the cancellation from the high frequency kernel of P^v_j, which gives the exponential gain $2^{-2\gamma j}$.

Let ζ be a smooth function on $\mathbf{R}$, with

$$\chi_{(-2,2)} \leq \hat{\zeta} \leq \chi_{(-3,3)}, \tag{4.13}$$

and set $\zeta_j(y) = 2^j \zeta(2^j y)$. We subtract

$$\int \phi_s(x - te^{\perp})\zeta_{j-4}(t)dt \tag{4.14}$$

from the term ϕ_s. Then by the fact that

$$P_j^v[\int \phi_s(x - te^{\perp})\zeta_{j-4}(t)dt] = 0 \tag{4.15}$$

we have that

$$\sum_{s\in\mathcal{AT}(2^k), scl(s)=scl} \langle f, \varphi_s\rangle P_j^v(\phi_s)$$
$$= \sum_{s\in\mathcal{AT}(2^k), scl(s)=scl} \langle f, \varphi_s\rangle P_j^v[\phi_s - \int \phi_s(x - te^{\perp})\zeta_{j-4}(t)dt],$$

hence it suffices to prove

$$\| \sum_{s\in\mathcal{AT}(2^k), scl(s)=scl} \langle f, \varphi_s\rangle (\phi_s - \int \phi_s(x - te^{\perp})\zeta_{j-4}(t)dt)\|_2 \lesssim 2^{-\gamma j}\|f\|_2. \tag{4.16}$$

By Lemma 4.3, we have the following pointwise estimate

$$|\phi_s - \int \phi_s(x - te^{\perp})\zeta_{j-4}(t)dt| \tag{4.17}$$

$$\lesssim (scl \cdot 2^{\eta j})^{-1+\epsilon}\chi_{R_s}^{(2)}(x)\chi_{\tilde{\omega}_s}(v(x)) + (scl \cdot 2^j)^{-10}\chi_{R_s}^{(2)}(x) + |R_s|^{-1/2}\chi_{F_s}(x) \tag{4.18}$$

where the set $F_s \subset v^{-1}(\tilde{\omega}_s)$ satisfies the estimate

$$|F_s| \lesssim 2^{-(1+\eta-\epsilon)j}|R_s| \tag{4.19}$$

with $\epsilon < \frac{\eta}{20}$ and $\chi_{R_s}^{(2)}$ the L^2 normalized function associated to R_s.

Then similar calculation as before finishes the proof of (4.16). So far we have finished the verification of the almost orthogonality condition.

References

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