A BILINEAR POLYNOMIAL CARLESON OPERATOR

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Abstract. We prove the boundedness of a bilinear polynomial Carleson operator. This is a question raised by Muscalu during the AIM workshop "Carleson theorems and multilinear operators" in May 2015.

1. Introduction

Take an integer $d \geq 2$. Consider

$$B(f,g) := \sup_{N_{\alpha,\beta}\in\mathbb{R}: 2\leq \alpha,\beta\leq d} \left| \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-t) g(x-s) e^{i\sum_{2\leq\alpha,\beta\leq d} N_{\alpha,\beta} t^\alpha s^\beta} K_0(t,s) dt ds \right|. \tag{1.1}$$

Here $K_0 : \mathbb{R}^2 \to \mathbb{R}$ is a Calderon-Zygmund kernel that satisfies the Hörmander-Mikhlin condition. This is a bilinear version of the polynomial Carleson operator introduced and studied by Stein and Wainger. We prove

Theorem 1.1. *For each $d \geq 2$, we have*

$$\|B(f,g)\|_r \lesssim \|f\|_p \|g\|_q, \tag{1.2}$$

for every

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \text{ with } 1 < p, q < \infty. \tag{1.3}$$

The implicit constant depends on p, q, r and the degree d.

Remark 1.1. By a straightforward generalisation of the argument below, one can also prove the full range, including the endpoint case

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \text{ with } 1 < p, q \leq \infty \text{ and } 0 < r < \infty. \tag{1.4}$$

The case when polynomials are linear was solved by Li and Muscalu [LM].

2. Proof

We first linearise our maximal operator and write it as

$$B(f,g) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-t) g(x-s) e^{i\sum_{2\leq\alpha,\beta\leq d} N_{\alpha,\beta}(x) t^\alpha s^\beta} K_0(t,s) dt ds. \tag{2.1}$$

Here $N_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ is a measurable function for each $2 \leq \alpha, \beta \leq d$.

For each $j \in \mathbb{Z}$, define

$$l^j(x) := \sup\{n \in \mathbb{Z} : 2^{\alpha n + \beta(n-j)} |N_{\alpha,\beta}(x)| \leq 1 \text{ for every } 2 \leq \alpha, \beta \leq d\}. \tag{2.2}$$

We write our operator as

$$\sum_{j\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-t) g(x-s) e^{i\sum_{2\leq\alpha,\beta\leq d} N_{\alpha,\beta}(x) t^\alpha s^\beta} K_0(t,s) \varphi_l(t) \varphi_{l-j}(s) dt ds. \tag{2.3}$$

Here

$$\varphi_l(t) := \varphi_0(\frac{t}{2^l}) \tag{2.4}$$

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and φ_0 is a non-negative function supported on $(1/2, 3) \cup (-3, -1/2)$ and is appropriately chosen such that

$$\sum_{l \in \mathbb{Z}} \varphi_l(t) = 1 \text{ for every } t \neq 0. \tag{2.5}$$

This is a product decomposition of our Calderon-Zygmund kernel, which might seem unusual at the first glance. However, having both t and s localised in dyadic intervals will allow us to treat the phase function $\sum_{2 \leq \alpha,\beta \leq d} N_{\alpha,\beta}(x) t^\alpha s^\beta$ very efficiently. Split (2.3) into two sums

$$\begin{aligned} &\sum_{j \in \mathbb{Z}} \sum_{l \leq l_j(x)} \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-t) g(x-s) e^{i \sum_{2 \leq \alpha,\beta \leq d} N_{\alpha,\beta}(x) t^\alpha s^\beta} K_0(t,s) \varphi_l(t) \varphi_{l-j}(s) dt ds \\ &+ \sum_{j \in \mathbb{Z}} \sum_{l \geq l_j(x)} \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-t) g(x-s) e^{i \sum_{2 \leq \alpha,\beta \leq d} N_{\alpha,\beta}(x) t^\alpha s^\beta} K_0(t,s) \varphi_l(t) \varphi_{l-j}(s) dt ds. \end{aligned} \tag{2.6}$$

We call them I and II separately.

We start with the term I, and compare it with

$$\tilde{I} := \sum_{j \in \mathbb{Z}} \sum_{l \leq l_j(x)} \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-t) g(x-s) K_0(t,s) \varphi_l(t) \varphi_{l-j}(s) dt ds \tag{2.7}$$

The difference can be easily bounded by the Hardy-Littlewood maximal operator. To be precise,

$$|I - \tilde{I}| \lesssim \sum_{j \in \mathbb{Z}} \sum_{l \leq l_j(x)} \int_{\mathbb{R}} \int_{\mathbb{R}} |f|(x-t) |g|(x-s) |K_0(t,s)| \varphi_l(t) \varphi_{l-j}(s) \left| \sum_{2 \leq \alpha,\beta \leq d} N_{\alpha,\beta}(x) t^\alpha s^\beta \right| dt ds \tag{2.8}$$

The Hörmander-Mikhlin condition tells us that

$$|K_0(t,s)| \lesssim \frac{1}{t^2 + s^2}. \tag{2.9}$$

This further implies

$$|I - \tilde{I}| \lesssim Mf(x) Mg(x). \tag{2.10}$$

Hence Hölder's inequality, together with the boundedness of the Hardy-Littlewood maximal operator, leads to

$$\|Mf Mg\|_r \lesssim \|f\|_p \|g\|_q, \text{ for } (p,q,r) \text{ satisfying (1.3).} \tag{2.11}$$

Hence to control term I, it remains to control $\tilde{I}$. By losing a constant, it suffices to bound

$$\sum_{j \in \mathbb{N}} \sum_{l \leq l_j(x)} \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-t) g(x-s) K_0(t,s) \varphi_l(t) \varphi_{l-j}(s) dt ds \tag{2.12}$$

For simplicity, we still call it $\tilde{I}$. We need the following simple lemma.

Lemma 2.1. *For every $x \in \mathbb{R}$, we have*

$$l_j(x) \leq l_{j+1}(x) \leq l_j(x) + 1, \text{ for every } j. \tag{2.13}$$

Moreover, we can not have $l_j(x) = l_{j+1}(x)$ for too many consecutive j, that is

$$l_j(x) < l_{j+100d} \text{ for every } j. \tag{2.14}$$

Similarly, we can not have $l_j(x) < l_{j+1}(x)$ for too many consecutive j, that is

$$l_{j+100d} < l_j(x) + 100d \text{ for every } j. \tag{2.15}$$

Proof. The proof of this lemma follows simply from the definition of the function l_j. We leave it out. □

Lemma 2.1 allows us to write $\tilde{I}$ as

$$\begin{aligned}&\sum_{j\in\mathbb{N}}\sum_{l\le l_0(x)}\int_{\mathbb{R}}\int_{\mathbb{R}} f(x-t)g(x-s)K_0(t,s)\varphi_l(t)\varphi_{l-j}(s)dtds\\ &+\sum_{k\in\mathbb{N}}\sum_{l_k(x)<l\le l_{k+1}(x)}\sum_{j\le l_{k+1}(x)-k-1}\int_{\mathbb{R}}\int_{\mathbb{R}} f(x-t)g(x-s)K_0(t,s)\varphi_l(t)\varphi_j(s)dtds\end{aligned} \tag{2.16}$$

By the estimate of maximal paraproducts due to Do, Muscalu and Thiele [DMT] (they indeed proved a much stronger variation-norm estimate), we have that

$$\left\|\sum_{j\in\mathbb{N}}\sum_{l\le l_0(x)}\int_{\mathbb{R}}\int_{\mathbb{R}} f(x-t)g(x-s)K_0(t,s)\varphi_l(t)\varphi_{l-j}(s)dtds\right\|_r \lesssim \|f\|_p\|g\|_q, \tag{2.17}$$

for every (r,p,q) satisfying (1.3). Moreover, by the decay property of the kernel K given by (2.9), it is not difficult to see that

$$\left|\sum_{l_k(x)<l\le l_{k+1}(x)}\sum_{j\le l_{k+1}(x)-k-1}\int_{\mathbb{R}}\int_{\mathbb{R}} f(x-t)g(x-s)K_0(t,s)\varphi_l(t)\varphi_j(s)dtds\right| \lesssim 2^{-\gamma k}Mf(x)Mg(x). \tag{2.18}$$

Here and in the rest, γ denotes a positive constant that may vary from line to line. By the triangle inequality, Hölder's inequality and bounds of the maximal operator, we obtain the desired bound.

It remains to control II. By the triangle inequality, it suffices to prove

$$\begin{aligned}&\left\|\int_{\mathbb{R}}\int_{\mathbb{R}} f(x-t)g(x-s)e^{i\sum_{2\le\alpha,\beta\le d}N_{\alpha,\beta}(x)t^\alpha s^\beta}K_0(t,s)\varphi_{l_j(x)+l}(t)\varphi_{l_j(x)+l-j}(s)dtds\right\|_r\\ &\lesssim 2^{-\gamma\max\{l,j\}}\|f\|_p\|g\|_q.\end{aligned} \tag{2.19}$$

It is not difficult to see that

$$\begin{aligned}&\left\|\int_{\mathbb{R}}\int_{\mathbb{R}} f(x-t)g(x-s)e^{i\sum_{2\le\alpha,\beta\le d}N_{\alpha,\beta}(x)t^\alpha s^\beta}K_0(t,s)\varphi_{l_j(x)+l}(t)\varphi_{l_j(x)+l-j}(s)dtds\right\|_r\\ &\lesssim 2^{-\gamma j}\|f\|_p\|g\|_q.\end{aligned} \tag{2.20}$$

Hence it suffices to consider

$$l \ge 100d^2 j, \tag{2.21}$$

which from now on we assume. Moreover, the estimate (2.19) without the decaying factor $2^{-\gamma\max\{l,j\}}$ is trivial. Hence by interpolation, it suffices to consider the case $(r,p,q)=(2,4,4)$.

Apply a change of variables

$$t\to 2^{l_j(x)+l}t,\ s\to 2^{l_j(x)+l-j}s. \tag{2.22}$$

The term we need to bound becomes

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(x-2^{l_j(x)+l}t)g(x-2^{l_j(x)+l-j}s)e^{i\sum_{2\le\alpha,\beta\le d}\tilde{N}_{\alpha,\beta}(x)t^\alpha s^\beta}K_0(2^{l_j(x)+l}t,2^{l_j(x)+l-j}s)\varphi_0(t)\varphi_0(s)dtds \tag{2.23}$$

multiplied by $2^{l_j(x)+l}2^{l_j(x)+l-j}$. Here

$$\tilde{N}_{\alpha,\beta}(x) := N_{\alpha,\beta}(x)2^{\alpha(l_j(x)+l)}2^{\beta(l_j(x)+l-j)}. \tag{2.24}$$

Moreover, we have

$$\sum_{2\le\alpha,\beta\le d}|\tilde{N}_{\alpha,\beta}(x)| \gtrsim 2^{3l/2} \text{ for every } x\in\mathbb{R}. \tag{2.25}$$

Write

$$\sum_{2\leq\alpha,\beta\leq d}\tilde{N}_{\alpha,\beta}(x)t^\alpha s^\beta=\sum_{2\leq\alpha\leq d}P_\alpha(s)t^\alpha, \tag{2.26}$$

where P_α is a polynomial of s, for each α. Then

$$\sum_{2\leq\alpha\leq d}\|P_\alpha\|_{l^1}\geq 2^{3l/2}. \tag{2.27}$$

Here for a polynomial P, we use $\|P\|_{l^1}$ to denote the l^1 norm of its coefficients. In the expression (2.23), if one does not mind assuming K_0 to be homogeneous, then it becomes

$$2^{-j}\int_{\mathbb{R}}\int_{\mathbb{R}}f(x-2^{l_j(x)+l}t)g(x-2^{l_j(x)+l-j}s)e^{i\sum_{2\leq\alpha,\beta\leq d}\tilde{N}_{\alpha,\beta}(x)t^\alpha s^\beta}K_0(t,2^{-j}s)\varphi_0(t)\varphi_0(s)dtds \tag{2.28}$$

In the end, we will gain a factor $2^{-\gamma l}$. Recall that we have assumed l to be much bigger compared with j. hence one can ignore the leading factor 2^{-j}. Now we start to estimate (2.28). If we denote

$$E_l:=\{s\in[0,2]:|P_\alpha(s)|\leq 2^{\frac{l}{10d}}\text{ for every }2\leq\alpha\leq d\}, \tag{2.29}$$

then

$$|E_l|\lesssim 2^{-\gamma l}\text{ for some }\gamma>0. \tag{2.30}$$

Split (2.28) into two terms

$$\begin{aligned}&\int_{\mathbb{R}}\Big[\int_{\mathbb{R}}f(x-2^{l_j(x)+l}t)e^{i\sum_{2\leq\alpha,\beta\leq d}\tilde{N}_{\alpha,\beta}(x)t^\alpha s^\beta}K_0(t,2^{-j}s)\varphi_0(t)dt\Big]\\&\times g(x-2^{l_j(x)+l-j}s)\varphi_0(s)\chi_{E_l}(s)ds\\&+\int_{\mathbb{R}}\Big[\int_{\mathbb{R}}f(x-2^{l_j(x)+l}t)e^{i\sum_{2\leq\alpha,\beta\leq d}\tilde{N}_{\alpha,\beta}(x)t^\alpha s^\beta}K_0(t,2^{-j}s)\varphi_0(t)dt\Big]\\&\times g(x-2^{l_j(x)+l-j}s)\varphi_0(s)\chi_{E_l^c}(s)ds\end{aligned} \tag{2.31}$$

To continue, we introduce the "small" maximal operator. For $\epsilon>0$, define

$$\mathcal{M}_\epsilon f(x):=\sup_{a>0;E\subset[-2,2],|E|\leq\epsilon}\int_{\mathbb{R}}|f|(x-t)\left(\frac{1}{a}\chi_E(\frac{t}{a})\right)dt. \tag{2.32}$$

For the first term in (2.31), we have

$$Mf(x)\mathcal{M}_{2^{-\gamma l}}g(x). \tag{2.33}$$

By Hölder, its L^r norm can be further bounded by

$$\|Mf\|_p\|\mathcal{M}_{2^{-\gamma l}}g\|_q\lesssim 2^{-\gamma l}\|f\|_p\|g\|_q. \tag{2.34}$$

Again the constant γ in the last display may differ from the one in (2.33). For the second term on the right hand side of (2.31),

$$\begin{aligned}&\left(\int_{\mathbb{R}}\Big[\int_{\mathbb{R}}f(x-2^{l_j(x)+l}t)e^{i\sum_{2\leq\alpha,\beta\leq d}\tilde{N}_{\alpha,\beta}(x)t^\alpha s^\beta}K_0(t,2^{-j}s)\varphi_0(t)dt\Big]^2\varphi_0(s)\chi_{E_l^s}(s)ds\right)^{1/2}\\&\times\left(\int_{\mathbb{R}}|g(x-2^{l_j(x)+l-j}s)|^2\varphi_0(s)ds\right)^{1/2}\end{aligned} \tag{2.35}$$

Take the L^2 norm of the last expression, and apply Hölder,

$$\begin{aligned}&\left\|\left(\int_{\mathbb{R}}\Big[\int_{\mathbb{R}}f(x-2^{l_j(x)+l}t)e^{i\sum_{2\leq\alpha,\beta\leq d}\tilde{N}_{\alpha,\beta}(x)t^\alpha s^\beta}K_0(t,2^{-j}s)\varphi_0(t)dt\Big]^2\varphi_0(s)\chi_{E_l^s}(s)ds\right)^{1/2}\right\|_4\\&\times\|M_2(g)\|_4\end{aligned} \tag{2.36}$$

Here M_2 is the Hardy-Littlewood maximal operator being applied on the square of a function. Apply Minkowski to the former part,

$$\sup_{s\in E_l^c}\left\|\int_{\mathbb{R}} f(x-2^{l_j(x)+l}t)e^{i\sum_{2\leq\alpha,\beta\leq d}\tilde{N}_{\alpha,\beta}(x)t^\alpha s^\beta}K_0(t,2^{-j}s)\varphi_0(t)dt\right\|_4. \tag{2.37}$$

Apply the main estimate in Stein-Wainger [SW],

$$\lesssim 2^{-\gamma l}\|f\|_4. \tag{2.38}$$

This implies the desired estimate.

References

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