Math 635 Introduction to Stochastic Calculus, Spring 2014
Homework 7 (Last One!)

Due 3 PM on Wednesday, April 30.

Generalities. Throughout these exercises $B_{\cdot} = \{B_t : t \in \mathbb{R}_+\}$ is standard Brownian motion.

1. First some definitions for complex-valued processes. The imaginary unit is $i = \sqrt{-1}$. Let $Z_t = U_t + iV_t$ be a complex-valued process with real part U_t and imaginary part V_t. Let us say that Z_t is a complex-valued local martingale if both U_t and V_t are local martingales.

If $f(\omega, t) = u(\omega, t) + iv(\omega, t)$ with $u, v \in \mathcal{L}^2_{\mathrm{LOC}}[0, T]$ we can define the stochastic integral as

$$\int_0^t f(\omega, s)\, dB_s(\omega) = \int_0^t u(\omega, s)\, dB_s(\omega) + i \int_0^t v(\omega, s)\, dB_s(\omega).$$

Here is the question. Let $f(\omega, t) = u(\omega, t) + iv(\omega, t)$ with $u, v \in \mathcal{L}^2_{\mathrm{LOC}}[0, T]$. Show that the process

$$M_t(\omega) = \exp\Big\{ \int_0^t f(\omega, s)\, dB_s(\omega) - \tfrac{1}{2} \int_0^t f(\omega, s)^2\, ds \Big\}$$

is a complex-valued local martingale.

2. Let $\lambda \in \mathbb{R}$. The exponential function $g(t) = e^{\lambda t}$ could be characterized as the unique continuous solution of the equation

$$g(t) = 1 + \lambda \int_0^t g(s)\, ds.$$

(a) Find the stochastic exponential, that is, the continuous solution X_t of the equation

$$X_t = 1 + \lambda \int_0^t X_s\, dB_s.$$

Show that there is a unique solution for this equation.

(b) Next, let Y_t be a standard process in Steele's terminology. Find the process Z_t that satisfies

$$Z_t = 1 + \lambda \int_0^t Z_s \, dY_s.$$

3. Exercise 10.1. from p. 167.

4. On the probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t\}$ let $\{f(\omega, t) : 0 \le t \le T\}$ be a bounded, adapted process and

$$X_t(\omega) = B_t(\omega) + \int_0^t f(\omega, s) \, ds, \qquad 0 \le t \le T.$$

Show that $P(a < X_t < b) > 0$ for all real $a < b$ and $t \in (0, T]$. (We can express this by saying that the *support* of the distribution of X_t is all of $\mathbb{R}$. By definition, the support of a probability distribution is the complement of the union of open sets of measure zero.)

Hint. Write $P(a < X_t < b) = E^P[\mathbf{1}_{(a,b)}(X_t)]$ and use Girsanov. Depending on how you argue, you may need this general fact: if $Y \ge 0$ and $E^P[Y] = 0$ then $P(Y = 0) = 1$.