Math 635 Introduction to Stochastic Calculus, Spring 2014
Homework 4

Due 3 PM on Monday, March 24. Note: if you do not intend to work during spring break, you need to do the problems this week!

Generalities. Throughout these exercises $B_\cdot = \{B_t : t \in \mathbb{R}_+\}$ is standard Brownian motion.

1. Let $f \in \mathcal{H}^2[0,T]$ and τ be a stopping time. Show that for $0 \le t \le T$,

$$\int_0^{\tau(\omega)\wedge t} f(\omega,s)\,dB_s(\omega) = \int_0^t f(\omega,s)\,\mathbf{1}\{s \le \tau(\omega)\}\,dB_s(\omega) \qquad \text{a.s.}$$

You can follow the outline below or produce your own proof using facts in the textbook.

(i) Let $\tau_n = 2^{-n}(\lfloor 2^n\tau \rfloor + 1)$. Check that τ_n is a stopping time and $\tau_n \searrow \tau$ as $n \nearrow \infty$. Check that $\tau \ge k/2^n$ iff $\tau_n \ge (k+1)/2^n$.

(ii) Let $\ell(n) = \lceil 2^n T \rceil$. Start with the telescoping sum

$$\int_0^{\tau_n \wedge t} f\,dB = \sum_{k=0}^{\ell(n)} \mathbf{1}\{\tau_n \ge \tfrac{k+1}{2^n}\}\left(\int_0^{\frac{k+1}{2^n}\wedge t} f\,dB \;-\; \int_0^{\frac{k}{2^n}\wedge t} f\,dB \right) \tag{1}$$

Use facts we have proved, such as $\int_0^t f(s)\,dB_s = \int_0^T \mathbf{1}_{(0,t]}(s)\,f(s)\,dB_s$ for $0 < t < T$, $\mathbf{1}_A \int_s^t f\,dB = \int_s^t \mathbf{1}_A f\,dB$ for $s < t$ and $A \in \mathcal{F}_s$, and linearity, to turn the right-hand side of (1) into $\int_0^t \mathbf{1}_{(0,\tau_n]}(s)\,f(s)\,dB_s$.

(iii) Show that

$$\int_0^t \mathbf{1}_{(0,\tau_n]}(s)\,f(s)\,dB_s \;\longrightarrow\; \int_0^t \mathbf{1}_{(0,\tau]}(s)\,f(s)\,dB_s \qquad \text{in } L^2\text{, as } n \to \infty.$$

(iv) Use path continuity on the left side of (1).

2. *The integral of a step function in* $\mathcal{L}^2_{loc}[0,T]$. Fix a partition $0 = t_0 < t_1 < \cdots < t_M = T$, and random variables $a_0, \dots, a_{M-1}$. Assume that a_i is almost surely finite and $\mathcal{F}_{t_i}$-measurable, but make no integrability assumptions on them. Define

$$g(\omega, s) = \sum_{i=0}^{M-1} a_i(\omega) \mathbf{1}_{(t_i, t_{i+1}]}(s).$$

The task is to show that $g \in \mathcal{L}^2_{\text{loc}}$ (virtually immediate) and that

$$\int_0^t g(s)\, dB_s = \sum_{i=0}^{M-1} a_i (B_{t_{i+1} \wedge t} - B_{t_i \wedge t})$$

as we would expect.

Hint. Here is one possible localizing sequence:

$$\nu_n(\omega) = \inf\{t : |g(\omega, t)| \geq n \text{ or } t \geq T\}.$$

Show that $g(\omega, t)\mathbf{1}\{t \leq \nu_n(\omega)\}$ is actually also a step function with the same partition. Then you know exactly what the approximating integral

$$X_{n,t}(\omega) = \int_0^t g(\omega, s)\, \mathbf{1}\{s \leq \nu_n(\omega)\}\, dB_s(\omega)$$

looks like, and the definition in Section 7.1 can be applied.

3. The next two problems are together only because both are short problems.

(a) Prove that the random variable

$$X = \int_0^T (B_s + s)\, dB_s$$

is in L^2, and compute its mean and variance.

(b) Find an expression for

$$\int_0^t (\cos B_s)\, dB_s$$

that does not involve any stochastic integrals.