

Math 635 Introduction to Stochastic Calculus, Spring 2014  
Homework 2

**Due 3 PM on Monday, Feb. 17.**

1. Suppose  $X \in L^2(P)$ , in other words that  $E[X^2] < \infty$ . Show that  $E(X | \mathcal{G})$  is the best approximation of  $X$  in the mean square sense, among all  $\mathcal{G}$ -measurable square-integrable random variables. In other words, show that

$$E[(X - Y)^2] \geq E[(X - E(X | \mathcal{G}))^2]$$

for every  $\mathcal{G}$ -measurable  $Y \in L^2(P)$ . *Hint:* Begin with the left hand side, add and subtract  $E(X | \mathcal{G})$  inside the square. Don't worry about the  $L^2(P)$  membership assumptions. Those are just made to guarantee that expectations of squares are finite.

2. Suppose  $B_t$  is a standard Brownian motion on some probability space. Let  $Y_t = B_{a+t} - B_a$  for some fixed (nonrandom)  $a > 0$ .

(a) Check that  $Y_t$  is a standard Brownian motion.

(b) Check that  $\{Y_t : t \geq 0\}$  is independent of  $\{B_t : t \in [0, a]\}$ . *Hint:* Some measure-theoretic technicalities (something called the  $\pi$ - $\lambda$  Theorem, you can take this step for granted) reduce this to checking that, for any  $0 \leq s_1 < \dots < s_m \leq a$  and  $0 \leq t_1 < \dots < t_n < \infty$ , the finite-dimensional vectors  $(B_{s_1}, \dots, B_{s_m})$  and  $(Y_{t_1}, \dots, Y_{t_n})$  are independent. You can do this for example by checking that the characteristic function splits into a product:

$$E[e^{i \sum_{k=1}^m \alpha_k B_{s_k} + i \sum_{j=1}^n \theta_j Y_{t_j}}] = E[e^{i \sum_{k=1}^m \alpha_k B_{s_k}}] \cdot E[e^{i \sum_{j=1}^n \theta_j Y_{t_j}}]$$

for all  $\alpha_k, \theta_j \in \mathbb{R}$ .

3. From p. 41 in the book, exercises 3.3 (a), (b) and (d). In (d), assume that the covariance matrix is nonsingular. Interpret the question as asking for the conditional density of  $Y$ , given that  $X = x$ .

4. Let  $f$  be a continuous function on  $[0, b]$  and  $B_t$  standard Brownian motion on some probability space  $(\Omega, \mathcal{F}, P)$ . Define

$$Y(\omega) = \int_0^b f(t) B_t(\omega) dt.$$

Identify the distribution of the random variable  $Y$ .

*Hints:* Calculate the characteristic function  $Ee^{i\theta Y}$ . Since both  $f(t)$  and  $B_t(\omega)$  are continuous in  $t$  (consider only such  $\omega$ ) the integral can be written as a limit of Riemann sums. Brownian increments are the most convenient to work with, so try to write things in terms of increments.

5. Exercise 3.4 from p. 42 in the book. If you have never used the moment generating function  $\psi(t) = Ee^{tX}$  to calculate moments, observe what happens when you differentiate repeatedly and then set  $t = 0$ . Do not worry about justifying differentiation inside the expectation.

6. Exercise 4.6 from p. 59 in the book. To prepare for this, you might look at how we applied a martingale to study the hitting time for random walk, and then read Section 4.5 to see how similar things are done for Brownian motion. That  $P(\tau < \infty) = 1$  is in the book on p. 55 so you can take that for granted. Do the exercise in three parts as follows. If you cannot do part (a), assume the conclusion of part (a) and do as much as you can of the rest.

(a) Show that  $B_\tau$  and  $\tau$  are independent, and also

$$P(B_\tau = A) = P(B_\tau = -A) = 1/2.$$

This does not need a long proof, but there is a trick. Here is a hint. Let  $X_t = -B_t$ , and define

$$\sigma = \inf\{t \geq 0 : X_t = A \text{ or } X_t = -A\}.$$

Since  $X_t$  is again a standard Brownian motion, the pairs  $(B_\tau, \tau)$  and  $(X_\sigma, \sigma)$  have the same distribution. Moreover,  $\tau$  and  $\sigma$  are actually equal.

(b) Calculate  $\phi(\lambda)$  by using the martingale.

(c) Find  $E(\tau^2)$  from  $\phi$ . You may run into a thicket of algebraic and calculus details. If you persevere and get to the correct answer, it should be  $5A^4/3$ .