POINTWISE CONVERGENCE OF LACUNARY SPHERICAL MEANS

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Abstract. This is an exposition of a model case of results in [6]. We show that if f is locally in $L \log\log L$ then the lacunary spherical means converge almost everywhere.

1. Introduction. Let f be a locally integrable function on $\mathbb{R}^d$ where $d \geq 2$. For any integer k let $\mathcal{A}_k f(x)$ be the spherical average of f over the sphere of radius 2^k in $\mathbb{R}^d$ centered at x; i.e.

$$\mathcal{A}_k f(x) = \int f(x + 2^k y) d\theta(y);$$

here $d\theta$ denotes the normalized Lebesgue measure on the unit sphere. Clearly we have $\lim_{k\to-\infty} \mathcal{A}_k f(x) = f(x)$ for all x if f is continuous everywhere. Moreover by results of C. Calderón [1] and Coifman and Weiss [4] we have

$$\lim_{k\to-\infty} \mathcal{A}_k f(x) = f(x) \quad \text{almost everywhere} \tag{1.1}$$

if f is locally in L^p for $p > 1$. It is well known [7] that such results are equivalent with a weak type (p,p) bound for the local maximal function $\mathcal{M} f(x) = \sup_{k\leq 0} |\mathcal{A}_k f(x)|$ and the above mentioned authors showed that the maximal operator is bounded on L^p. It is still unknown whether (1.1) holds for $f \in L^1$ and, equivalently, whether the maximal function is of weak type (1,1). However we have

Theorem. *Let $\mathcal{M}$ be the global lacunary spherical maximal operator defined by*

$$\mathcal{M} f(x) = \sup_{k\in\mathbb{Z}} |\mathcal{A}_k f(x)|. \tag{1.2}$$

There is a constant C so that for all measurable functions f and all $\alpha > 0$ the inequality

$$\big|\{x \in \mathbb{R}^n : \mathcal{M} f(x) > \alpha\}\big| \leq \int \frac{C|f(x)|}{\alpha} \log\log\big(e^2 + \frac{C|f(x)|}{\alpha}\big) dx \tag{1.3}$$

holds.

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As a corollary one obtains the pointwise convergence result (1.1) if f belongs locally to $L \log\log L$.

The inequality for the lacunary spherical maximal function is a special case of more general results in [6] which apply to operators such as maximal averages and Hilbert transforms along plane curves which are homogeneous with respect to some family of a nonisotropic dilations. The presence of the nonisotropic dilation structure causes additional difficulties and therefore it seems adequate to present the less technical proof of the theorem above separately. The main idea from [6] is still present.

Concerning weak type inequalities for classes near L^1 we mention two previous results for the operator $\mathcal{M}$. First Christ and Stein [3] showed (combining Calderón-Zygmund arguments with Yano's extrapolation theorem) that $\mathcal{M}$ maps $L\log L(Q_0)$ (for a unit cube Q_0) to Weak L^1. This result applies to more general maximal operators associated to Borel measures, whose Fourier transform decays at infinity. Moreover Christ [2] showed the harder result that $\mathcal{M}$ maps the Hardy space $H^1(\mathbb{R}^d)$ to Weak L^1 (for weak type (p,p) endpoint bounds for related maximal operators see also the recent paper [5]). The condition $f \in H^1$ means that f has some rather substantial cancellation. Concerning size estimates note there is a restriction if $f \in H^1$ is single signed in an open ball; namely then f belongs to $L\log L(K)$ for all compact subsets K of this ball (*cf.*[8, §I.5.2 (c)]).

Notation. For two quantities A and B we write $A \lesssim B$ or $B \gtrsim A$ if there exists an absolute positive constant C so that $A \le CB$. The Lebesgue measure of a set E will be denoted by $|E|$.

2. Length and thickness. We say that a set E is *granular* if E is a finite union of dyadic cubes. Our argument will center around a certain decomposition of an arbitrary granular set E into sets of various "length" and "thickness". We shall first give definitions of these quantities.

For a granular set E we define a dyadic version of a *one-dimensional Hausdorff content* or simply *"length"* $\lambda(E)$ to be

$$\lambda(E) := \inf_{\mathcal{Q}} \sum_{Q \in \mathcal{Q}} l(Q) \tag{2.1}$$

where $\mathcal{Q}$ ranges over all finite collections $\mathcal{Q}$ of dyadic cubes with $E \subset \cup_{Q\in\mathcal{Q}} Q$, and $l(Q)$ denotes the side-length of Q.

Next if E is granular, we define the *thickness* $\Theta(E)$ to be

$$\Theta(E) := \sup_{Q} \frac{|E \cap Q|}{l(Q)} \tag{2.2}$$

where Q ranges over all dyadic cubes. Clearly, if E is contained in a dyadic cube q it is sufficient to let Q in (2.2) range over all dyadic subcubes of q.

The quantities of length and thickness are complementary. For instance, it is immediate from the definitions that one has

$$|E| \le \lambda(E)\Theta(E). \tag{2.3}$$

The bound (2.3) can be attained, for instance if E is a dyadic box. More generally if $C \ge 1$, we call E a *generalized box* with admissible deviation C if one has

$$\lambda(E)\Theta(E) \le C|E|. \tag{2.4}$$

In the proof of the weak type $L \log\log L$ inequality the quantity $\lambda(E)$ will control the size of the exceptional set, while $\Theta(E)$ will control the L^2 norm of the maximal function outside of the exceptional set. Inequalities such as (2.4) will be crucial for balancing the two estimates. There is also an intermediate range of scales in which neither of two quantities $\lambda(E)$ or $\Theta(E)$ is favorable, and one will just use L^1 estimates for that portion.

Now unfortunately (2.4) is not always satisfied but the following proposition can be used to efficiently decompose a granular set into generalized boxes of different lengths.

Proposition. *Let E be a granular set. Then there exists a decomposition $E = F \cup G$ into disjoint granular subsets F, G such that*

$$\lambda(F) \leq \frac{1}{2}\lambda(E) \tag{2.5}$$

and

$$\Theta(G) \leq 8\frac{|G|}{\lambda(E)}. \tag{2.6}$$

In particular, G is a generalized box with admissible deviation of at most 8.

Proof. Fix E and define the *critical thickness* $\vartheta(E)$ to be the largest non-negative number r such that the inequality

$$r\lambda(E) \leq 2r \sum_{Q\in\mathcal{Q}} l(Q) + |E \backslash \bigcup_{Q\in\mathcal{Q}} Q| \tag{2.7}$$

holds for all finite collections $\mathcal{Q}$ of dyadic cubes (here the empty collection is admitted). Equivalently, one can define $\vartheta(E)$ by

$$\vartheta(E) := \inf_{\mathcal{Q}} \frac{|E\backslash \bigcup_{Q\in\mathcal{Q}} Q|}{(\lambda(E) - 2\sum_{Q\in\mathcal{Q}} l(Q))_+}. \tag{2.8}$$

Clearly $\vartheta(E) \leq |E|/\lambda(E)$. Observe also that $\vartheta(E) > 0$. This follows because $|E\backslash \bigcup_{Q\in\mathcal{Q}} Q|$ is bounded away from zero whenever $\sum_{Q\in\mathcal{Q}} l(Q) \leq \lambda(E)/2$ (thanks to the hypothesis that E is granular). Moreover, again since E is granular, there exists a finite collection $\mathcal{Q}_1$ of dyadic cubes such that

$$\vartheta(E)\lambda(E) = 2\vartheta(E) \sum_{Q\in\mathcal{Q}_1} l(Q) + |E_*| \tag{2.9}$$

where E_* is the set

$$E_* := E\backslash \bigcup_{Q\in\mathcal{Q}_1} Q. \tag{2.10}$$

We claim that

$$\Theta(E_*) \leq 2\vartheta(E). \tag{2.11}$$

Indeed, suppose that there existed a cube Q' such that

$$|E_* \cap Q'| > 2\vartheta(E)l(Q'). \tag{2.12}$$

Then $Q' \notin \mathcal{Q}_1$. If we apply (2.7) to the collection $\mathcal{Q}_1 \cup \{Q'\}$ we obtain

$$\vartheta(E)\lambda(E) \le 2\vartheta(E)(l(Q') + \sum_{Q\in\mathcal{Q}_1} l(Q)) + |E_*| - |E_* \cap Q'|,$$

but this contradicts (2.9) and (2.12). This proves (2.11).

One might now try to set G equal to E_* and F equal to $E \setminus E_*$. While one can control the length of $E \setminus E_*$ and the thickness of E_* the problem is that $|E_*|$ could be very small. To deal with this we need to fatten E_* up, and to do this we use a recursive construction to obtain large subsets of E of bounded thickness.

Lemma. *Let $r > 0$. For any dyadic cube I, there exists a (possibly empty) collection $\mathcal{Q}[I]$ of disjoint dyadic cubes in I and a granular set $E[I] \subset E \cap I$ such that*

$$\Theta(E[I]) \le 2r \tag{2.13}$$

and

$$2|E[I]| \ge 2r \sum_{Q\in\mathcal{Q}[I]} l(Q) + |(E\cap I)\setminus \bigcup_{Q\in\mathcal{Q}[I]} Q|. \tag{2.14}$$

The Lemma is proved by induction on the sidelength of I. If $l(I) \le (2r)^{\frac{1}{d-1}}$, the lemma follows simply by taking $E[I] = E \cap I$ and $\mathcal{Q}[I]$ to be empty. Now fix a dyadic cube I and suppose that the lemma has been proven for all proper dyadic subcubes I'. Partition I into 2^d sub-cubes $I_1, \dots, I_{2^d}$ of side-length $\frac{1}{2}l(I)$. By the inductive hypothesis, we may construct collections $\mathcal{Q}[I_j]$ and sets $E[I_j]$ for $j = 1, \dots, 2^d$ satisfying the properties of the lemma. We distinguish two cases. First if $|\bigcup_{j=1}^{2^d} E[I_j]| \le 2rl(I)$ then we define $E[I] := \bigcup_{j=1}^{2^d} E[I_j]$ and $\mathcal{Q}[I] := \bigcup_{j=1}^{2^d} \mathcal{Q}[I_j]$. Next if $|\bigcup_{j=1}^{2^d} E[I_j]| > 2rl(I)$ then we take $E[I]$ to be a granular subset of $\bigcup_{j=1}^{2^d} E[I_j]$ of measure at least $rl(I)$, and at most $2rl(I)$ and take $\mathcal{Q}[I]$ to be the singleton set $\{I\}$. The properties (2.13/14) are not hard to check in both cases (see also [6] for a more detailed description of a variant).

Proof of the Proposition, cont. Since E is granular there is a dyadic cube q so that E is contained in it. We apply the Lemma with $I = q$ and $r = \vartheta(E)$. We thus find a set $E[q]$ and a collection $\mathcal{Q}[q]$ obeying the properties in the lemma. We now set $G := E_* \cup E[q]$ and $F := E\setminus G$.

To show (2.5) we observe $F \subset E\setminus E_* \subset \bigcup_{Q\in\mathcal{Q}_1} Q$, so $\lambda(F) \le \sum_{Q\in\mathcal{Q}_1} l(Q)$. But by (2.9) this sum must be less than or equal to $\lambda(E)/2$, which gives (2.5).

To show (2.6) we first observe that

$$|G| \ge |E[q]| \ge \frac{1}{2}(2\vartheta(E)) \sum_{Q\in\mathcal{Q}[q]} l(Q) + \Big|E\setminus \bigcup_{Q\in\mathcal{Q}[q]} Q\Big|$$

by (2.14), since now $r = \vartheta(E)$. By (2.7) we thus see that $|G| \ge \lambda(E)\vartheta(E)/2$. Since $\Theta(G) \le \Theta(E_*)+\Theta(E[q]) \le 2\vartheta(E)+2\vartheta(E) = 4\vartheta(E)$, we see that $\Theta(G) \le 8|G|/\lambda(E)$ which is (2.6). $\square$

3. Basic reductions. We say that a function f is granular if $f = \sum_\nu c_\nu \chi_{E_\nu}$ where $c_\nu \in \mathbb{C}$, the sum is finite, the sets E_ν are granular and mutually disjoint. By a limiting argument it suffices to consider granular function in the proof of the theorem.

Let now f be a granular function and we shall estimate the size of $\{x : \mathcal{M}f(x) > \alpha\}$. We perform a standard Calderón-Zygmund decomposition at height 1 for the function $\Phi(|f|/\alpha)$, where $\Phi(t) = t \log\log(e^2 + t)$. This can be done via the Whitney decomposition theorem applied to the open set

$$\Omega = \{x : M_{HL}(\Phi(|f|/\alpha))(x) > 1\}$$

where M_{HL} is the Hardy-Littlewood maximal function. We denote by $\mathfrak{Q}$ the set of Whitney cubes arising in this fashion. By possibly subdividing each cube Q into cubes of length $2^{-10}l(Q)$ we may assume that $2^8 l(Q) \le d^{-1/2}\mathrm{dist}(Q, \mathbb{R}^d \setminus \Omega_\alpha) \le 2^{10} l(Q)$. From this it is easy to see that we may subdivide the family $\mathfrak{Q}$ into subcollections $\mathfrak{Q}_1, \dots, \mathfrak{Q}_{N(d)}$ with the property that in each $\mathfrak{Q}_i$ the double cubes Q^* are pairwise disjoint.

We shall slightly modify the definition of our maximal operator. Let $d\sigma_k$ be the measure given by

$$\langle f, d\sigma_k \rangle = \int f(2^k y)\chi(y) d\theta(y)$$

where χ is supported on a ball of radius $\le 1/2$ (so that the support of $d\sigma_k$ is contained in the sphere of radius 2^k centered at the origin, but does not contain antipodal points on this sphere). We only need to consider the maximal operator M given by $Mf(x) = \sup_{k\in\mathbb{Z}} |d\sigma_k * f(x)|$.

Now let $g(x) = f(x)$ if $|f(x)| \le 2^{10}\alpha$ and $g(x) = 0$ otherwise. Then $f - g$ is supported in Ω. Since $|g(x)| \lesssim \alpha$ the L^2 boundedness of the maximal operator and Chebyshev's inequality can be used to show that

$$\Big|\{x : |Mg(x)| > \alpha/2\}\Big| \lesssim \int \frac{|f(x)|}{\alpha} dx \lesssim \int \Phi(\frac{|f(x)|}{\alpha}) dx. \tag{3.1}$$

Also if $\widetilde{\Omega}$ denotes the union of the tenfold expanded cubes then

$$|\widetilde{\Omega}| \lesssim |\Omega| \lesssim \int \Phi(\frac{|f(x)|}{\alpha}) dx \tag{3.2}$$

and thus it suffices to show that

$$\Big|\{x \notin \widetilde{\Omega} : |M[f-g](x)| > \alpha/2\}\Big| \lesssim \int \Phi(\frac{|f(x)|}{\alpha}) dx; \tag{3.3}$$

(3.1), (3.2), (3.3) imply the assertion of the Theorem. In order to prove (3.3) we split the function $f - g$ further. For $n = 10, 11, \dots$ let

$$E^n = \{x \in \Omega : 2^n\alpha < |f| \le 2^{n+1}\alpha\}$$

and let $f^n(x) = f(x)\chi_{E^n}(x)$. Now $E^n = \cup_{q\in\mathfrak{Q}} E^n_q$ where $E^n_q = E^n \cap q$. Note that the sets E^n_q are granular since f was assumed to be a granular function. We use Proposition 2.1 iteratively to decompose E^n_q further, namely

$$E^n_q = \bigcup_{\nu=0}^{\infty} E^{n,\nu}_q \cup F^n_q$$

where F_q^n has measure zero and will be ignored in what follows, and where each $E_q^{n,\nu}$ is a generalized box, the sets $E_q^{n,\nu}$ are mutually disjoint, and $\lambda(E_q^n \setminus \cup_{\nu=1}^m E_q^{n,\nu}) \le 2^{-m}\lambda(E_q^n)$. Thus also

$$|E_q^{n,\nu}| \le \lambda(E_q^{n,\nu}) l(q)^{d-1} \le 2^{1-\nu}|q|. \tag{3.4}$$

Finally define

$$f_q^{n,\nu}(x) = f(x)\chi_{E_q^{n,\nu}}(x). \tag{3.5}$$

We shall first handle the terms with $\nu \ge n^2$ and show that

$$\Big|\{x : \sup_k |\sum_q \sum_n \sum_{\nu \ge n^2} f_q^{n,\nu} * d\sigma_k(x)| > \alpha/4\}\Big| \lesssim \int \Phi(\frac{|f(x)|}{\alpha}) dx. \tag{3.6}$$

In view of (3.4) these terms are again easy to handle by an L^2 estimate. By Chebyshev's inequality and the L^2 boundedness of the maximal operator we get

$$\Big|\{x : \sup_k |\sum_q \sum_n \sum_{\nu \ge n^2} f_q^{n,\nu} * d\sigma_k(x)| > \alpha/4\}\Big| \lesssim \alpha^{-2}\Big\|\sum_q \sum_n \sum_{\nu \ge n^2} f_q^{n,\nu}\Big\|_2^2$$
$$\lesssim \alpha^{-2}\int \Big|\sum_q \sum_n \sum_{\nu \ge n^2} 2^n \alpha \chi_{E_q^{n,\nu}}(x)\Big|^2 dx \lesssim \sum_q \sum_n \sum_{\nu \ge n^2} 2^{2n}|E_q^{n,\nu}|$$

by the disjointness of the sets $E_q^{n,\nu}$. By (3.4) the last expression is bounded by a constant times

$$\sum_q \sum_n \sum_{\nu \ge n^2} 2^{2n} 2^{-\nu}|q| \lesssim \sum_q |q| \lesssim \int \Phi(\frac{|f(x)|}{\alpha}) dx$$

which yields (3.6).

We are left with the consideration of terms with $\nu < n^2$ in the complement of the set $\widetilde{\Omega}$. Since for $2^k \le l(q)$ the convolution $d\sigma_k * f_q^{n,\nu}$ is supported in $\widetilde{\Omega}$ we are reduced to verify that

$$\Big|\{x : \sup_k |\sum_{\substack{q: \\ 2^k > l(q)}} \sum_n \sum_{\nu=1}^{n^2} f_q^{n,\nu} * d\sigma_k(x)| > \alpha/4\}\Big| \lesssim \int \Phi(\frac{|f(x)|}{\alpha}) dx. \tag{3.7}$$

This will be carried out in the next section. Clearly (3.6) and (3.7) imply the desired estimate (3.3).

4. Proof of (3.7). For each n, ν, q let $k_q^{n,\nu}$ be unique integer for which

$$2^{k_q^{n,\nu}-1} < \max\big\{l(q), \big(2^n \log(10+n)\Theta(E_q^{n,\nu})\big)^{1/(d-1)}\big\} < 2^{k_q^{n,\nu}}. \tag{4.1}$$

We consider the contribution to the case $k \le k_q^{n,\nu}$. This contribution is supported inside the set $\mathcal{V} = \widetilde{\Omega} \cup \mathcal{V}_1$ where

$$\mathcal{V}_1 = \bigcup_q \bigcup_n \bigcup_\nu \bigcup_{k: l(q) \le 2^k \le 2^{k_q^{n,\nu}}} (E_q^{n,\nu} + S_k) \tag{4.2}$$

and $E_q^{n,\nu}+S_k$ is the Minkowski sum of the set $E_q^{n,\nu}$ and the sphere S_k of radius 2^k.

By covering $E_q^{n,\nu}$ efficiently by cubes in q (*cf.* the definition of λ) we see that the inner union has measure at most

$$\lesssim \lambda(E_q^{n,\nu})\Theta(E_q^{n,\nu})2^n\log(10+n)\approx |E_q^{n,\nu}|2^n\log(1+n)$$

since $E_q^{n,\nu}$ is a generalized box. Thus

$$|\mathcal{V}|\lesssim|\widetilde{\Omega}|+\sum_q\sum_n\sum_\nu|E_q^{n,\nu}|2^n\log(10+n)\lesssim\int\Phi\Big(\frac{|f(x)|}{\alpha}\Big)dx \tag{4.3}$$

by the disjointness and definition of the sets $E_q^{n,\nu}$.

Next let (for $n\ge 10$)

$$\kappa_n=100\log_2 n \tag{4.4}$$

and we consider the contribution of the scales

$$k_q^{n,\nu}<k\le k_q^{n,\nu}+\kappa_n. \tag{4.5}$$

For this case we replace the sup by the sum and use Chebyshev's inequality in L^1 to estimate

$$\begin{aligned}
&\mathrm{meas}\Big(\{x:\sup_k\sum_{n,\nu}\sum_{\substack{q:k_q^{n,\nu}<k\\ \le k_q^{n,\nu}+\kappa_n}}|f_q^{n,\nu}*d\sigma_k(x)|>\alpha/8\}\Big)\\
&\le 8\alpha^{-1}\Big\|\sup_k\sum_{n,\nu}\sum_{\substack{q:k_q^{n,\nu}<k\le\\ k_q^{n,\nu}+\kappa_n}}f_q^{n,\nu}*d\sigma_k\Big\|_1\le 8\alpha^{-1}\sum_q\sum_{n,\nu}\sum_{\substack{k:k_q^{n,\nu}<k\le\\ k_q^{n,\nu}+\kappa_n}}\|f_q^{n,\nu}\|_1\\
&\lesssim\sum_{n,\nu}\log\log(10+2^n)\sum_q\int_q 2^n\chi_{E_q^{n,\nu}}(x)dx\lesssim\int\Phi\Big(\frac{|f(x)|}{\alpha}\Big)dx.
\end{aligned}\tag{4.6}$$

For the remainder, we shall actually show that

$$\mathrm{meas}\Big(\Big\{x:\sup_k\Big|\sum_q\sum_{\substack{n,\nu:\\ k>k_q^{n,\nu}+\kappa_n\\ \nu<n^2}}f_q^{n,\nu}*d\sigma_k(x)\Big|>\alpha/8\Big\}\Big)\lesssim\alpha^{-1}\int|f(x)|dx \tag{4.7}$$

and the right hand side is of course controlled by $\int\Phi(|f(x)|/\alpha)dx$. Clearly the desired estimate (3.7) follows from (4.3), (4.6) and (4.7).

Introducing cancellation. As in standard Calderón-Zygmund theory we modify the functions $f_q^{n,\nu}$ to introduce some cancellation. Namely let $\{P_i\}_{i=1}^M$ be an orthonormal basis of the space of polynomials of degree $\le 100d$ on the unit cube

$[-1/2,1/2]^d$ and for a cube q with center x_q and length $l(q)$ define the projection operator Π_q by

$$\Pi_q[f](x)=\chi_q(x)\sum_{i=1}^{M}P_i\big(\frac{x-x_q}{l(q)}\big)\int_q f(y)P_i\big(\frac{y-x_q}{l(q)}\big)\frac{dy}{l(q)^d}.$$

Note that

$$|\Pi_q[h](x)|\le C\frac{1}{|q|}\int_q|h(y)|dy \tag{4.8}$$

where C is independent of h and q.

Let

$$b_q^{n,\nu}(x)=f_q^{n,\nu}(x)-\Pi_q[f_q^{n,\nu}](x)$$

so that $b_q^{n,\nu}$ vanishes off q and

$$\int_q b_q^{n,\nu}(x)x^\alpha dx=0 \qquad \text{if } |\alpha|\le 100d. \tag{4.9}$$

We observe that since the q's are Whitney cubes for Ω, we have

$$\sum_{n,\nu}\big|\Pi_q f_q^{n,\nu}(x)\big|\lesssim\chi_q(x)\frac{1}{|q|}\int_q|f(x)|dx\lesssim\alpha; \tag{4.10}$$

moreover by (4.8)

$$\sum_{n,\nu}\|b_q^{n,\nu}\|_1\lesssim\sum_{n,\nu}\|f_q^{n,\nu}\|_1\lesssim\int_q|f(x)|dx. \tag{4.11}$$

Now (4.7) will follow by Chebyshev's inequality from

$$\Big\|\sup_k\Big|\sum_q\sum_{\substack{n,\nu:\\ k>k_q^{n,\nu}+\kappa_n\\ \nu<n^2}}\Pi_q[f_q^{n,\nu}]*d\sigma_k\Big|\Big\|_2^2\lesssim\alpha\int|f(x)|dx \tag{4.12}$$

$$\Big\|\sup_k\Big|\sum_q\sum_{\substack{n,\nu:\\ k>k_q^{n,\nu}+\kappa_n\\ \nu<n^2}}b_q^{n,\nu}*d\sigma_k\Big|\Big\|_2^2\lesssim\alpha\int|f(x)|dx \tag{4.13}$$

since $\alpha\|f\|_1\lesssim\alpha^2\int\Phi(f/|\alpha)dx$.

From (4.10) and the disjointness of the cubes q we have $\sum_{q,n,\nu}\big|\Pi_q f_q^{n,\nu}(x)\big|\lesssim\alpha$ and the estimate (4.12) is immediate because of the positivity and L^2 boundedness of the lacunary spherical maximal operator, and (4.10).

For the remainder of the paper we prove (4.13). We replace the sup by an ℓ^2 norm (in k) and use Minkowski's inequality to estimate the left hand side of (4.13) by

$$\Big[\sum_{i=1}^{N(d)}\Big(\sum_k\Big\|\sum_{q\in\mathfrak{Q}_i}\sum_{\substack{n,\nu:\nu<n^2\\ k>k_q^{n,\nu}+\kappa_n}}b_q^{n,\nu}*d\sigma_k\Big\|_2^2\Big)^{1/2}\Big]^2\lesssim\sum_{i=1}^{N(d)}(I_i+II_i)$$

where

(4.14)
$$I_i = \sum_k \sum_{q\in\mathfrak{Q}_i} \Big|\langle \widetilde{d\sigma_k} * d\sigma_k * \sum_{\substack{n,\nu:\\ k>k_q^{n,\nu}+\kappa_n\\ \nu<n^2}} b_q^{n,\nu}, \sum_{\substack{n',\nu':\\ k>k_q^{n',\nu'}+\kappa_{n'}\\ \nu'<n'^2}} b_q^{n',\nu'}\rangle\Big|$$

(4.15)
$$II_i = 2\sum_k \sum_{\substack{q,q'\in\mathfrak{Q}_i\\ q\neq q'\\ l(q)\le l(q')}} \Big| \sum_{\substack{n,\nu:\\ k>k_q^{n,\nu}+\kappa_n\\ \nu<n^2}} \sum_{\substack{n',\nu':\\ k>k_{q'}^{n',\nu'}+\kappa_n'\\ \nu'<n'^2}} \langle \widetilde{d\sigma_k} * d\sigma_k * b_q^{n,\nu}, b_{q'}^{n',\nu'}\rangle\Big|;$$

and we shall consider separately the terms I_i and II_i.

In order to estimate these expressions we use the following well known estimate

$$\big|\partial^\gamma\big[\widetilde{d\sigma_k} * d\sigma_k\big](x)\big| \lesssim 2^{-k(d-1)}|x|^{-1-|\gamma|}\chi_{|x|\le 2^{k+1}} \tag{4.16}$$

for all multiindices $\gamma\in\mathbb{N}_0^d$; here we need the assumption on the small support of $d\sigma_0$.

The cancellation of the functions $b_q^{n,\nu}$ will only play a role for the estimation of II_i; here no geometric information on the sets $E_q^{n,\nu}$ is used. We carry out this estimate and use the moment conditions of order $N=10d$ on the $b_q^{n,\nu}$, the fact that the cubes in $\mathfrak{Q}_i$ are separated and the estimate (4.16).

Since the doubly expanded cubes are disjoint by construction of the family $\mathfrak{Q}_i$ we obtain for $l(q)<l(q')$, $x\in q'$,

$$|\widetilde{d\sigma_k} * d\sigma_k * b_q^{n,\nu}(x)| \lesssim 2^{-k(d-1)} l(q)^N |x-x_q|^{-N-1}\|b_q^{n,\nu}\|_1.$$

Thus

$$II_i \lesssim \sum_{n,\nu}\sum_{q\in\mathfrak{Q}_i}\|b_q^{n,\nu}\|_1 \sum_{\substack{q'\in\mathfrak{Q}_i\\ l(q')\ge l(q)\\ q'\neq q}} \sum_{k>l(q)} 2^{-(d-1)k} l(q)^N \operatorname{dist}(q,q')^{-N-1}\sum_{n',\nu'}\|b_{q'}^{n',\nu'}\|_1$$

Next note that by (4.11)

$$\sum_{n',\nu'}\int_{q'}|b_{q'}^{n',\nu'}(y)|dy \lesssim \int_{q'}|f(y)|dy \lesssim \alpha|q'|.$$

Moreover, we have for fixed q that $\operatorname{dist}(q,q')\gtrsim l(q')$ if $q',q\in\mathfrak{Q}_i, l(q')\ge l(q)$ and $q\neq q'$. Thus

$$\sum_{\substack{q'\in\mathfrak{Q}_i\\ l(q')\ge l(q)\\ q'\neq q}} \sum_{k>l(q)} 2^{-(d-1)k} l(q)^N \operatorname{dist}(q,q')^{-N-1}|q'|$$
$$\lesssim l(q)^{N-d+1}\int_{|x-x_q|\ge l(q)}|x-x_q|^{-N-1}dx \lesssim 1.$$

Combining the two previous estimates and applying (4.11) again yields

$$II_i \lesssim \alpha\sum_{n,\nu}\sum_q \|b_q^{n,\nu}\|_1 \lesssim \alpha\|f\|_1. \tag{4.17}$$

Estimation of the main term I_i. We estimate $I_i \le I_{i,1} + I_{i,2}$ where

$$
\begin{aligned}
I_{i,1} &= \sum_k \sum_{q\in\mathfrak{Q}_i} \Big| \Big\langle \widetilde{d\sigma_k} * d\sigma_k * \sum_{\substack{n,\nu:\\ k>k_q^{n,\nu}+\kappa_n\\ \nu<n^2}} f_q^{n,\nu}, \sum_{\substack{n',\nu':\\ k>k_q^{n',\nu'}+\kappa_{n'}\\ \nu'<n'^2}} b_q^{n',\nu'} \Big\rangle \Big| \\
I_{i,2} &= \sum_k \sum_{q\in\mathfrak{Q}_i} \Big| \Big\langle \widetilde{d\sigma_k} * d\sigma_k * \sum_{\substack{n,\nu:\\ k>k_q^{n,\nu}+\kappa_n\\ \nu<n^2}} \Pi_q[f_q^{n,\nu}], \sum_{\substack{n',\nu':\\ k>k_q^{n',\nu'}+\kappa_{n'}\\ \nu'<n'^2}} b_q^{n',\nu'} \Big\rangle \Big|
\end{aligned}
\tag{4.18}
$$

The estimation of $I_{i,2}$ is rather straightforward. By (4.16) for $\gamma = 0$ and by (4.10/11) we get

$$
\begin{aligned}
I_{i,2} &\lesssim \sum_q \int_q \sum_{k\ge l(q)} 2^{-k(d-1)} \int_q \frac{1}{|x-y|}\alpha\chi_q(y)dy \,\Big| \sum_{\substack{n',\nu':\\ k>k_q^{n',\nu'}+\kappa_{n'}\\ \nu'\le n'^2}} b_q^{n',\nu'}(x)\Big| dx \\
&\lesssim \alpha \sum_q \sum_{k\ge l(q)} (2^{-k}l(q))^{d-1} \sum_{\substack{n',\nu':\\ k>k_q^{n',\nu'}+\kappa_{n'}\\ \nu'\le n'^2}} \int_q |b_q^{n',\nu'}(x)|dx \lesssim \alpha\|f\|_1.
\end{aligned}
\tag{4.19}
$$

Now we estimate the more substantial term $I_{i,1}$ and use the estimation in terms of the thickness of the sets $E_q^{n,\nu}$.

Using (4.16) with $\gamma = 0$ we bound

$$
I_{i,1} \lesssim \sum_k \sum_{q\in\mathfrak{Q}_i} \sum_{\substack{n',\nu':\\ k>k_q^{n',\nu'}+\kappa_{n'}}} \int |b_q^{n',\nu'}(x)| \sum_{n\ge 10} \big[A_{n,}^{k,q}(x) + B_n^{k,q}(x)\big]dx
$$

where

$$
\begin{aligned}
A_n^{k,q}(x) &= \int\limits_{\substack{2^{-2n}l(q)\le\\ |x-y|\le l(q)}} \frac{2^{-k(d-1)}}{|x-y|} 2^n\alpha \sum_{\substack{\nu<n^2:\\ k>k_q^{n,\nu}+\kappa_n}} \chi_{E_q^{n,\nu}}(y)dy,\\
B_n^{k,q}(x) &= \int\limits_{\substack{|x-y|\le\\ 2^{-2n}l(q)}} \frac{2^{-k(d-1)}}{|x-y|} 2^n\alpha \sum_{\substack{\nu<n^2:\\ k>k_q^{n,\nu}+\kappa_n}} \chi_{E_q^{n,\nu}}(y)dy.
\end{aligned}
$$

The estimate for the terms involving $B_n^{k,q}(x)$ is straightforward. Since $2^{k_q^{n,\nu}} \ge l(q)$ we get

$$
B_n^{k,q}(x) \lesssim \int_{|x-y|\le 2^{-2n}l(q)} 2^n\alpha \frac{2^{-k(d-1)}}{|x-y|}dy \Big\| \sum_{\nu<n^2} \chi_{E_q^{n,\nu}} \Big\|_\infty \lesssim \alpha 2^{-n(2d-3)}(2^{-k}l(q))^{d-1}
$$

and thus

$$\sum_k \sum_{q\in\mathfrak{Q}_i} \sum_{\substack{n',\nu':\\ k>k_q^{n',\nu'}+\kappa_{n'}}} \int |b_q^{n',\nu'}(x)| \sum_{n\ge 10} B_n^{k,q}(x)dx$$

$$\lesssim \alpha \sum_q \sum_{n',\nu'} \int |b_q^{n',\nu'}(x)|dx \lesssim \alpha \|f\|_1.$$

Next for the main term we use that for $x \in q$

$$\int_{\substack{2^{-n+m-1}l(q)<\\ |x-y|\le 2^{-n+m}l(q)}} \frac{1}{|x-y|}\chi_{E_q^{n,\nu}}(y)dy \lesssim \Theta(E_q^{n,\nu})$$

and therefore

$$A_n^{k,q}(x) \lesssim 2^{-k(d-1)}2^n\alpha \sum_{\substack{\nu\le n^2\\ k>k_q^{n,\nu}+\kappa_n}} 2n\Theta(E_q^{n,\nu}) \lesssim 2n\alpha \sum_{\substack{\nu\le n^2\\ k>k_q^{n,\nu}+\kappa_n}} 2^{-(k-k_q^{n,\nu})(d-1)}.$$

Now we perform the k summation and since $2^{-\kappa_n} = n^{-100}$ we get

$$\sum_k \sum_{q\in\mathfrak{Q}_i} \sum_{\substack{n',\nu':\\ k>k_q^{n',\nu'}+\kappa_{n'}}} \int |b_q^{n',\nu'}(x)| \sum_{n\ge 10} A_n^{k,q}(x)dx$$

$$\lesssim \sum_{n'} \sum_{q\in\mathfrak{Q}_i} \sum_{n',\nu'} \int |b_q^{n',\nu'}(x)|dx \sum_{n\ge 10} n^{3-100(d-1)}\alpha \lesssim \alpha\|f\|_1.$$

Putting the estimates together we obtain $I_i \lesssim \alpha\|f\|_1$ and by (4.17) the expression II_i satisfies the same bound. This yields the desired estimate (4.13) and finishes the proof. □

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