ON RADIAL AND CONICAL FOURIER MULTIPLIERS

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ABSTRACT. We investigate connections between radial Fourier multipliers on $\mathbb{R}^d$ and certain conical Fourier multipliers on $\mathbb{R}^{d+1}$. As an application we obtain a new weak type endpoint bound for the Bochner-Riesz multipliers associated to the light cone in $\mathbb{R}^{d+1}$, where $d \geq 4$, and results on characterizations of $L^p \to L^{p,\nu}$ inequalities for convolutions with radial kernels.

INTRODUCTION

This paper is a sequel to [9] in which the authors obtained a characterization of radial multipliers of $\mathcal{F}L^p(\mathbb{R}^d)$ provided that $1 < p < 2$ and the dimension d is large enough. The main estimate in [9] was concerned with a convolution inequality for surface measure on spheres which in this paper we state as Hypothesis $\mathrm{Sph}(p_1, d)$ for some $p_1 > 1$. Under this hypothesis we shall prove several equivalent statements on cone multipliers and radial Fourier multipliers.

In what follows we fix a radial $C^\infty(\mathbb{R}^d)$ function $\psi_\circ$ supported in a small ball centered at the origin (say, of radius $\leq (100d)^{-1}$) whose Fourier transform vanishes at the origin to high order (say $100d$). We assume that $\widehat{\psi_\circ}(\xi) \neq 0$ for $1/8 \leq |\xi| \leq 8$. Set

$$\psi = \psi_\circ * \psi_\circ \tag{0.1}$$

and, for $y \in \mathbb{R}^d$ and for $r \geq 1$, let σ_r be surface measure on the sphere of radius r centered at the origin, i.e.

$$\langle \sigma_r, f \rangle = r^{d-1} \int_{S^{d-1}} f(ry') d\sigma_1(y') \,. \tag{0.2}$$

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Hypothesis $\mathbf{Sph(p,d)}$. *There is a constant C so that for every $h \in L^p(\mathbb{R}^d \times \mathbb{R}^+; dy\, r^{d-1}dr)$ the inequality*

$$(0.3)\quad \Big\| \int_{\mathbb{R}^d}\int_1^\infty h(y,r)\sigma_r * \psi(\cdot - y)\, dr dy \Big\|_{L^p(\mathbb{R}^d)} \le C\Big(\iint_{\mathbb{R}^d\times\mathbb{R}^+} |h(y,r)|^p dy\, r^{d-1}dr\Big)^{1/p}$$

holds.

Theorem 0.1. *([9]) Hypothesis Sph(p,d) holds for $d \ge 4$, $1 \le p < \frac{2(d-1)}{d+1}$.*

1. Statement of results

In what follows $L^{p,\nu}$ denotes the standard Lorentz space, and we shall usually assume that $p \le \nu \le \infty$. We denote by $\mathcal{F}_d f$ the $\mathbb{R}^d$ Fourier transform of f, defined by $\mathcal{F}_d f(\xi) = \int f(y) e^{-i\langle y,\xi\rangle} dy$. We shall also write $\mathcal{F}f$ or $\widehat{f}$ if the dimension is clear from the context.

Let $\gamma = \{\gamma_k\}_{k\in\mathbb{Z}}$ be a sequence of bounded functions supported in $(-1/4, 1/4)$. Define

$$(1.1)\qquad m(\xi,\tau) \equiv m_\gamma(\xi,\tau) = \sum_{k\in\mathbb{Z}} \gamma_k\big(\frac{|\xi|-\tau}{2^k}\big)\, \mathbb{1}_{[2^k,2^{k+1})}(\tau)$$

where $\mathbb{1}_E$ denotes the characteristic function of E. Let T denote the operator acting on Schwartz functions in $\mathbb{R}^{d+1}$ by

$$(1.2)\qquad \mathcal{F}_{d+1}[Tf](\xi,\tau) = m_\gamma(\xi,\tau)\mathcal{F}_{d+1}f(\xi,\tau).$$

Moreover, for each fixed $\tau \in (0,\infty)$, define an operator T^τ on functions in $\mathbb{R}^d$ by

$$(1.3)\qquad \mathcal{F}_d[T^\tau f](\xi) = \gamma_k\big(\frac{|\xi|-\tau}{2^k}\big)\mathcal{F}_d f(\xi), \qquad \text{if } \tau \in [2^k, 2^{k+1}).$$

Theorem 1.1. *Let T, T^τ be as in (1.2), (1.3).*

Suppose that $1 < p_1 < \frac{2d}{d+1}$ and suppose that Hypothesis Sph(p_1, d) holds. Let $1 < p < p_1$, $p \le \nu \le \infty$. Then the following statements are equivalent.

(i) T maps $L^p(\mathbb{R}^{d+1})$ boundedly to $L^{p,\nu}(\mathbb{R}^{d+1})$.

(ii) There is a constant C_p so that for all sequences $\{\tau_k\}_{k=-\infty}^\infty$ satisfying $\tau_k \in [2^k, 2^{k+1})$, and for all $f \in L^p(\mathbb{R}^d)$

$$\Big\| \sum_{k\in\mathbb{Z}} \alpha_k T^{\tau_k} f\Big\|_{L^{p,\nu}(\mathbb{R}^d)} \le C_p \sup_{k\in\mathbb{Z}} |\alpha_k|\, \|f\|_{L^p(\mathbb{R}^d)}$$

(iii) For every $k \in \mathbb{Z}$ there is a $\tau_k \in [2^k, 2^{k+1})$ such that T^{τ_k} maps $L^p(\mathbb{R}^d)$ boundedly to $L^{p,\nu}(\mathbb{R}^d)$, and such that $\sup_k \|T^{\tau_k}\|_{L^p \to L^{p,\nu}} < \infty$.

(iv) The functions $s \mapsto \widehat{\gamma}_k(s)\,(1+|s|)^{-\frac{d-1}{2}}$ belong to the weighted Lorentz space $L^{p,\nu}(\mathbb{R}, (1+|\cdot|)^{d-1})$, with the uniform bound

$$\sup_{k\in\mathbb{Z}} \left\| \frac{\widehat{\gamma}_k}{(1+|\cdot|)^{\frac{d-1}{2}}} \right\|_{L^{p,\nu}(\mathbb{R},(1+|r|)^{d-1}dr)} < \infty\,. \tag{1.4}$$

(v) The functions $\mathcal{F}_d^{-1}[\gamma_k(|\cdot|)]$ belong to $L^{p,\nu}(\mathbb{R}^d)$, with the uniform bound

$$\sup_{k\in\mathbb{Z}} \big\| \mathcal{F}_d^{-1}[\gamma_k(|\cdot|)] \big\|_{L^{p,\nu}(\mathbb{R}^d)} < \infty\,.$$

From Theorem 0.1 we get

Corollary 1.2. *Statements (i)-(v) in Theorem 1.1 are equivalent if $d \ge 4$, $1 < p < \frac{2(d-1)}{d+1}$, $p \le \nu \le \infty$.*

The equivalence of (iv) $\iff$ (v) and the implication (iii) $\implies$ (iv) are in [5]. The implication (ii) $\implies$ (iii) is trivial. The implication (i) $\implies$ (iii) follows from a version of de Leeuw's theorem, see Lemma 2.3. It is not presently clear how to deduce the global statement (ii) directly from (i), without going through (iv) or (v). The proofs of the main implications (iv) $\implies$ (i) and (iv) $\implies$ (ii) are given in §4, §5; they rely on Hypothesis Sph(p_1, d).

As a consequence of the implication (iv) $\implies$ (i) we shall derive a new endpoint result for the so-called Bochner-Riesz multipliers for the cone, defined by

$$\rho_\lambda(\xi,\tau) = \Big(1 - \frac{|\xi|^2}{\tau^2}\Big)_+^\lambda. \tag{1.5}$$

It is conjectured that ρ_λ is a multiplier of $\mathcal{F}L^p(\mathbb{R}^{d+1})$ if $\lambda > d(1/p-1/2)-1/2$ and $1 < p < \frac{2d}{d+1}$; this condition is necessary. This conjecture is open in the full p-range. The first sharp L^p results for some range of p were proved by T. Wolff [19] in two dimensions, with extensions and improvements in [13], [4], [6], [7], [9]. For the endpoint $\lambda = d(1/p-1/2)-1/2$ one conjectures a weak type (p,p) inequality for $p < \frac{2d}{d+1}$. This endpoint inequality cannot be replaced by a stronger statement such as $L^p \to L^{p,\nu}$ boundedness for $\nu < \infty$. In §6 we prove

Corollary 1.3. *Let $d \ge 2$ and $p_1 > 1$, and suppose that Hypothesis Sph(p_1, d) holds. Let ρ_λ be as in (1.5). If $\lambda = d(1/p-1/2)-1/2$ and $1 < p < p_1$ then*

$$\big\| \mathcal{F}^{-1}[\rho_\lambda \widehat{f}] \big\|_{L^{p,\infty}(\mathbb{R}^{d+1})} \le C_p \|f\|_{L^p(\mathbb{R}^{d+1})} \tag{1.6}$$

for all $f \in L^p(\mathbb{R}^{d+1})$. *In particular,* (1.6) *holds for* $d \geq 4$ *and* $1 < p < \frac{2(d-1)}{d+1}$.

Remark. Sharp bounds in H^p, $p < 1$ and sharp bounds for the operator acting on functions of the form $f(x,t) = f_0(|x|,t)$ can be found in Hong's articles [10], [11]. More recently, Heo, Hong and Yang [8] proved a weak type $(1,1)$ inequality for a localized cone multiplier $\chi(\tau)\rho_{(d-1)/2}(\xi,\tau)$, in dimension $d \geq 4$. As a corresponding result for the global cone multiplier one can prove that for $\mathrm{Re}\,(\lambda) = (d-1)/2$ the operator $f \to \mathcal{F}^{-1}_{d+1}[\rho_\lambda \widehat{f}]$ is bounded from the Hardy space H^1 to $L^{1,\infty}$, under the assumption that $Sph(p_1,d)$ holds for some $p_1 > 1$. This can be obtained by an analytic interpolation argument using the analytic family of multipliers $\lambda \to \rho_\lambda$, the $H^p \to L^{p,\infty}$ bounds in [10] for $p < 1$ and $\mathrm{Re}\,(\lambda) = d(1/p - 1/2) - 1/2$, and the L^p result of Corollary 1.3. For the justification of the analytic interpolation one uses an adaptation of arguments in [15]. We shall not give details here.

The equivalence of condition (ii) in the theorem with conditions (iv) or (v) immediately yields a generalization of the main result in [9] to $L^p \to L^{p,\nu}$ inequalities.

Corollary 1.4. *Let* $m = m_0(|\cdot|)$ *be a bounded radial function on* $\mathbb{R}^d$ *and define* $\mathcal{T}_m$ *by*

$$\mathcal{F}_d[\mathcal{T}_m f] = m\mathcal{F}_d f.$$

Let $1 < p_1 < \frac{2d}{d+1}$ *and assume that Hypothesis* $Sph(p_1,d)$ *holds. Let* $1 < p < p_1$, $p \leq \nu \leq \infty$. *Then, for any Schwartz function* $\eta \neq 0$

$$\big\|\mathcal{T}_m\big\|_{L^p \to L^{p,\nu}} \approx \sup_{t>0} t^{d/p} \big\|\mathcal{T}_m[\eta(t\cdot)]\big\|_{L^{p,\nu}}. \tag{1.7}$$

Moreover, if $\phi \in C_c^\infty$ *is compactly supported in* $(0,\infty)$ *(and not identically zero) and* κ_t *denotes the Fourier transform on* $\mathbb{R}$ *of the function* $\phi m_0(t\cdot)$ *then*

$$\big\|\mathcal{T}_m\big\|_{L^p \to L^{p,\nu}} \approx \sup_{t>0} \Big\| \frac{\kappa_t}{(1+|\cdot|)^{\frac{d-1}{2}}} \Big\|_{L^{p,\nu}(\mathbb{R};(1+|r|)^{d-1}dr)} < \infty. \tag{1.8}$$

The equivalence of the two conditions on the right hand sides of (1.7) and (1.8) with $L^p_{rad} \to L^{p,\nu}$ boundedness (i.e. on radial functions, for $p < \frac{2d}{d+1}$) was proved in [5]. The L^p case $(p = \nu)$ for $1 < p < \frac{2(d-1)}{d+1}$ was proved in [9], moreover that article has already $L^p \to L^{p,\nu}$ inequalities for radial multipliers which are compactly supported away from the origin.

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2. Preliminaries

The following dyadic interpolation lemma is convenient in dealing with the short range estimation in §4; it is proved in §2 of [9].

Lemma 2.1. *Let $0 < p_0 < p_1 < \infty$. Let $\{F_j\}_{j\in\mathbb{Z}}$ be a sequence of measurable functions on a measure space $\{\Omega, \mu\}$, and let $\{s_j\}$ be a sequence of nonnegative numbers. Assume that, for all j, the inequality*

$$\|F_j\|_{p_\nu}^{p_\nu} \le 2^{jp_\nu} M^{p_\nu} s_j$$

holds for $\nu = 0$ and $\nu = 1$. Then for every $p \in (p_0, p_1)$, there is a constant $C = C(p_0, p_1, p)$ such that

$$\Big\| \sum_j F_j \Big\|_p^p \le C^p M^p \sum_j 2^{jp} s_j\,.$$

We need a simple fact about Lorentz spaces.

Lemma 2.2. *Let $(\mathcal{X}_1, \mu_1)$, $(\mathcal{X}_2, \mu_2)$ be σ-finite measure spaces, and let $\mu = \mu_1 \times \mu_2$ be the product measure on $\mathcal{X}_1 \times \mathcal{X}_2$. Then, for $1 \le p < \infty$, $p \le \nu \le \infty$, and any μ-measurable function G,*

$$\|G\|_{L^{p,\nu}(\mathcal{X}_1\times\mathcal{X}_2,\mu)} \le C_{p,\nu}\Big(\int \|G(x_1,\cdot)\|^p_{L^{p,\nu}(\mathcal{X}_2,\mu_2)} d\mu_1\Big)^{1/p}. \tag{2.1}$$

The proof is a Fubini-type argument (in conjunction with Minkowski's inequality in $\ell^{\nu/p}$), we refer to §9 of [9].

Finally we need a version of a restriction theorem for multipliers due to de Leeuw.

Lemma 2.3. *Let $1 < p < \infty$ and $1 \le \nu_1, \nu_2 \le \infty$ and let m be a bounded continuous function in $\mathbb{R}^{d+1}$. Suppose that the operator $f \mapsto \mathcal{F}^{-1}_{d+1}[m\mathcal{F}_{d+1} f]$ is bounded from L^{p,ν_1} to L^{p,ν_2} with operator norm B. Let, for $\xi \in \mathbb{R}^d$, $m_0(\xi) = m(\xi, 0)$. Then there is a constant C independent of m and f such that*

$$\big\|\mathcal{F}_d^{-1}[m_0\mathcal{F}_d f]\big\|_{L^{p,\nu_2}} \le CB\big\|f\big\|_{L^{p,\nu_1}}.$$

Proof. This is just a modification of the proof given in [12] for $p = \nu_1 = \nu_2$. By the hypothesis

$$\Big|\iint m(\xi,\tau)\widehat{F}(\xi,\tau)\widehat{G}(\xi,\tau)d\xi d\tau\Big| \le B\|F\|_{L^{p,\nu_1}}\|G\|_{L^{p',\nu_2'}}\,. \tag{2.2}$$

Now let χ be a Schwartz function on $\mathbb{R}$ whose Fourier transform is supported in $(-1, 1)$. so that $\widehat{\chi}(0) \ne 0$. Given a small parameter δ we let $\chi_\delta(t) = \chi(\delta t)$, and, for $f \in L^{p,\nu_1}(\mathbb{R}^d)$, $g \in L^{p',\nu_2'}(\mathbb{R}^d)$ we define $F_\delta(x,t) = \delta\chi_\delta(t)f(x)$ and $G_\delta(x,t) = \chi_\delta(t)g(x)$. Observe that the inequality

$$\|h \otimes \chi_\delta\|_{L^{q,r}(\mathbb{R}^{d+1})} \le C(\chi)\delta^{-1/q}\|h\|_{L^{q,r}(\mathbb{R}^{d+1})}$$

is immediate for $q = r$, by Fubini, and then holds for arbitrary r by real interpolation. Thus $\|F_\delta\|_{L^{p,\nu_1}} \le \delta^{1-1/p}\|f\|_{L^{p,\nu_1}}$ and $\|G_\delta\|_{L^{p',\nu_2'}} \le \delta^{-1/p'}\|g\|_{L^{p',\nu_2}}$. Apply (2.2) with F_δ, G_δ and let $\delta \to 0$. This yields

$$\begin{aligned}
&[\widehat{\chi}(0)]^2\Big|\int m(\xi,0)\widehat{f}(\xi)\widehat{g}(\xi)d\xi\Big| \\
&= \lim_{\delta\to 0}\Big|\iint m(\xi,\tau)\widehat{f}(\xi)\widehat{g}(\xi)\delta^{-1}[\widehat{\chi}(\delta^{-1}\tau)]^2 d\xi d\tau\Big| \\
&\le CB\|f\|_{L^{p,\nu_1}}\|g\|_{L^{p',\nu_2'}}
\end{aligned}$$

which implies the assertion. $\square$

3. Inequalities for spherical measures

We shall now derive a consequence of Hypothesis $\mathrm{Sph}(p_1,d)$ which will be used in conjunctions with atomic decompositions. Similar inequalities have been used in [9] but they were derived using the proof of (0.3) rather than (0.3) itself.

In what follows let ℓ be a nonnegative integer and, for $\mathfrak{z} = (z_1,\dots,z_d) \in \mathbb{Z}^d$, let

$$R_{\mathfrak{z},\ell} = \{x \in \mathbb{R}^d : 2^\ell z_i \le x_i < 2^\ell z_i + 2^\ell, i = 1,\dots,d\} \tag{3.1}$$

so that the $R_{\mathfrak{z},\ell}$ form a tiling of $\mathbb{R}^d$ with dyadic cubes of sidelength 2^ℓ. We denote by $\chi_{R_{\mathfrak{z},\ell}}$ the characteristic function of $R_{\mathfrak{z},\ell}$. We denote variables in $\mathbb{Z}^{d+1}$ by $z = (\mathfrak{z}, z_{d+1})$ with $\mathfrak{z} \in \mathbb{Z}^d$. Let $I_{z_{d+1},\ell} = [2^\ell z_{d+1}, 2^\ell z_{d+1} + 2^\ell)$ and let

$$\chi_{z,\ell}(x,t) := \chi_{R_{\mathfrak{z},\ell}}(x)\chi_{I_{z_{d+1},\ell}}(t). \tag{3.2}$$

For each $r > 0$, $z \in \mathbb{Z}^{d+1}$ we are given an $L^2(\mathbb{R}^{d+1})$ function $g_{z,r}$ depending continuously on r such that

$$\|g_{z,r}\|_{L^2(\mathbb{R}^{d+1})} \le 1, \qquad \text{for all } z, r. \tag{3.3}$$

Moreover, for each positive integer n we are given an L^1 function ω_n supported in $[1/2, 2]$ so that

$$\sup_n \int_{1/2}^2 |\omega_n(\rho)|d\rho \le 1. \tag{3.4}$$

Let $\ell > 0$. Given $|b| \le 2$ we define an operator $\mathcal{S}_{\ell,b}$ acting on functions F on $\mathbb{Z}^{d+1} \times \mathbb{R}^+$ by

$$\text{(3.5)} \quad \mathcal{S}_{\ell,b}F(x,t) = \sum_z \sum_{n=\ell}^{\infty} \int_{2^n}^{2^{n+1}} F(z,r) \int_{1/2}^{2} \omega_n(\rho) \int_{\mathbb{R}^d} \psi{*}\sigma_{\rho r}(x-y)[\chi_{z,\ell}\, g_{z,r}](y,t-br)\, dy\, d\rho\, dr.$$

On the set $\mathbb{Z}^{d+1} \times \mathbb{R}^+$ we define the measure μ_d by

$$\mu_d(E) = \int_0^\infty \sum_{z \in \mathbb{Z}^{d+1}:(z,r)\in E} r^{d-1} dr$$

for a measurable set $E \subset \mathbb{Z}^{d+1} \times \mathbb{R}^+$.

Proposition 3.1. *Suppose $d \ge 2$ and Hypothesis Sph(p_1, d) holds for some $p_1 \in (1,2)$. Let $g_{z,r}$, ω_n be as in (3.3), (3.4), $\ell > 0$, $|b| \le 2$, and define $\mathcal{S}_{\ell,b}$ by (3.5). Then the inequality*

$$\text{(3.6)} \qquad \big\|\mathcal{S}_{\ell,b}F\big\|_{L^{p,\nu}(\mathbb{R}^{d+1})} \le C_{p,\nu}\, 2^{\ell[(d+1)(\frac1p - \frac12)-\alpha]} \|F\|_{L^{p,\nu}(\mathbb{Z}^{d+1}\times\mathbb{R}^+,\, \mu_d)}$$

holds for (i) for $p = 1 = \nu$, with $\alpha = \frac{d-1}{2}$, (ii) for $p = p_1 = \nu$, with $\alpha = 0$ and, (iii), for

$$1 < p < p_1, \quad 0 < \nu \le \infty \quad \textit{with } \alpha \le \frac{d-1}{2} \frac{\frac1p - \frac1{p_1}}{1 - \frac1{p_1}}.$$

Proof. Statement (iii) follows by real interpolation from the cases $p = \nu = p_1$ and $p = \nu = 1$.

We consider the case $p = p_1 = \nu$. In order to apply Hypothesis Sph(p_1, d) we interchange the ρ- and the r-integrals and change variables $s = r\rho$. This yields

$$\mathcal{S}_{\ell,b}F(x,t) = \sum_z \sum_{n=\ell}^{\infty} \int_{\rho=1/2}^{2} \int_{s=2^n\rho}^{2^{n+1}\rho} F(z, \tfrac{s}{\rho})\omega_n(\rho) \times \int_{\mathbb{R}^d} \psi * \sigma_s(x-y)[\chi_{z,\ell}\, g_{z,\frac{s}{\rho}}](y, t - b\tfrac{s}{\rho})\, dy\, ds \frac{d\rho}{\rho}\,,$$

and thus

$$\text{(3.7)} \qquad \mathcal{S}_{\ell,b}F(x,t) = \int_{2^{\ell-1}}^{\infty} \int_{\mathbb{R}^d} \psi * \sigma_s(x-y) V_{\ell,b}F(y,s,t)\, dy\, ds$$

where

$$\text{(3.8)} \quad V_{\ell,b}F(y,s,t) := \int_{\rho=1/2}^{2} \sum_{n=\ell}^{\infty} \omega_n(\rho)\chi_{[2^n\rho, 2^{n+1}\rho]}(s) \sum_z F(z, \tfrac{s}{\rho})\, [\chi_{z,\ell}\, g_{z,\frac{s}{\rho}}](y, t - b\tfrac{s}{\rho})\rho^{-1} d\rho\,.$$

For fixed t we apply Hypothesis $\mathrm{Sph}(p_1,d)$ and then integrate in t. This yields

$$\|\mathcal{S}_{\ell,b}F\|_{L^{p_1}(\mathbb{R}^{d+1})}=\Big(\int_t\|\mathcal{S}_{\ell,b}F(\cdot,t)\|^{p_1}_{L^{p_1}(\mathbb{R}^d)}dt\Big)^{1/p_1}$$
$$\lesssim\Big(\int\int_{2^{\ell-1}}^\infty\int\big|V_{\ell,b}F(y,s,t)|^{p_1}dy\,s^{d-1}ds\,dt\Big)^{1/p_1}.$$

We observe that if $2^\nu<s<2^{\nu+1}$ then only the terms with $\nu-1\le n\le\nu+1$ contribute to the n-sum in (3.8). Thus, for fixed (y,t),

$$(3.9)\quad \Big(\int_{2^{\ell-1}}^\infty|V_{\ell,b}F(y,s,t)|^{p_1}s^{d-1}ds\Big)^{1/p_1}\le\sum_{i=-1,0,1}$$
$$\Big(\sum_{\nu=\ell-1}^\infty\int_{2^\nu}^{2^{\nu+1}}\Big|\int_{\rho=1/2}^2\omega_{\nu+i}(\rho)\sum_z F(z,\tfrac{s}{\rho})\,[\chi_{z,\ell}\,g_{z,\frac{s}{\rho}}](y,t-b\tfrac{s}{\rho})\frac{d\rho}{\rho}\Big|^{p_1}s^{d-1}ds\Big)^{1/p_1}.$$

Now we have for fixed ν

$$\Big(\int_{2^\nu}^{2^{\nu+1}}\Big|\int_{\rho=1/2}^2\omega_{\nu+i}(\rho)\sum_z F(z,\tfrac{s}{\rho})\,[\chi_{z,\ell}\,g_{z,\frac{s}{\rho}}](y,t-b\tfrac{s}{\rho})\rho^{-1}d\rho\Big|^{p_1}s^{d-1}ds\Big)^{1/p_1}$$
$$\le\int_{1/2}^2|\omega_{\nu+i}(\rho)|\Big(\int_{2^\nu}^{2^{\nu+1}}\Big|\sum_z F(z,\tfrac{s}{\rho})\,[\chi_{z,\ell}\,g_{z,\frac{s}{\rho}}](y,t-b\tfrac{s}{\rho})\Big|^{p_1}s^{d-1}ds\Big)^{1/p_1}\frac{d\rho}{\rho}$$
$$\le\int_{1/2}^2|\omega_{\nu+i}(\rho)|\rho^{\frac{d}{p_1}-1}d\rho\Big(\int_{2^{\nu-1}}^{2^{\nu+2}}\Big|\sum_z F(z,r)\,[\chi_{z,\ell}\,g_{z,r}](y,t-br)\Big|^{p_1}r^{d-1}dr\Big)^{1/p_1}$$
$$\lesssim\Big(\int_{2^{\nu-1}}^{2^{\nu+2}}\Big|\sum_z F(z,r)\,[\chi_{z,\ell}\,g_{z,r}](y,t-br)\Big|^{p_1}r^{d-1}dr\Big)^{1/p_1}.$$

We insert this back into (3.9) and obtain

$$\Big(\int_{2^{\ell-1}}^\infty|V_{\ell,b}F(y,s,t)|^{p_1}s^{d-1}ds\Big)^{1/p_1}$$
$$\lesssim\Big(\int_{2^{\ell-1}}^\infty\Big|\sum_z F(z,r)\,[\chi_{z,\ell}\,g_{z,r}](y,t-br)\Big|^{p_1}r^{d-1}dr\Big)^{1/p_1}$$
$$\le\Big(\int_{2^{\ell-1}}^\infty\sum_z\Big|F(z,r)\,[\chi_{z,\ell}\,g_{z,r}](y,t-br)\Big|^{p_1}r^{d-1}dr\Big)^{1/p_1}.$$

We take L^{p_1} norms in (y,t) and, for fixed r, perform a shear transformation to get

$$\Big(\int\int\int_{2^{\ell-1}}^\infty|V_{\ell,b}F(y,s,t)|^{p_1}s^{d-1}ds\,dt\,dy\Big)^{1/p_1}$$
$$\lesssim\Big(\int_{2^{\ell-1}}^\infty\sum_z|F(z,r)|^{p_1}\iint\Big|\chi_{z,\ell}\,g_{z,r}(y,t)\Big|^{p_1}dt\,dy\,r^{d-1}dr\Big)^{1/p_1}.$$

By Hölder's inequality and our normalizing assumption (3.3) this is estimated by

$$\Big(\int_{2^{\ell-1}}^{\infty}\sum_z |F(z,r)|^{p_1} 2^{\ell(d+1)(1-p_1/2)}\|\chi_{z,\ell}\, g_{z,r}\|_2^{p_1} r^{d-1}dr\Big)^{1/p_1}$$
$$\lesssim 2^{\ell(d+1)(1/p_1-1/2)}\Big(\int\sum_z |F(z,r)|^{p_1} r^{d-1}dr\Big)^{1/p_1}.$$

This yields the assertion for $p=p_1=\nu$, with $\alpha=0$.

For $p=1$ we estimate

$$\big\|\mathcal{S}_{\ell,b}F\big\|_{L^1(\mathbb{R}^{d+1})} \lesssim \sum_{n=\ell}^{\infty}\sum_z\int_{2^n}^{2^{n+1}}|F(z,r)|\int_t\int_{\rho=1/2}^{2}|\omega_n(\rho)|\times$$
$$\big\|\psi*\sigma_{r\rho}*[\chi_{R_{\mathfrak{z},\ell}}\, g_{z,r}](\cdot,t-br)\big\|_{L^1(\mathbb{R}^d)}\,d\rho\,\chi_{I_{z_{d+1},\ell}}(t-br)\,dt\,dr.$$

The function $\psi*\sigma_{r\rho}*[\chi_{R_{\mathfrak{z},\ell}} g_{z,r}](\cdot,t-br)$ is supported in a set of measure $\le C2^\ell r^{d-1}$, namely an annulus of width $\lesssim 2^\ell$ built on a sphere of radius $r\rho$. Moreover we have $\big\|\mathcal{F}_d[\psi*\sigma_{r\rho}]\big\|_\infty \le Cr^{\frac{d-1}{2}}$ where C is independent of $\rho\in[1/2,2]$. Thus the last displayed expression can be estimated by

$$\sum_z\sum_{n=\ell}^{\infty}\int_{2^n}^{2^{n+1}}|F(z,r)|\int_t\int_{\rho=1/2}^{2}|\omega_n(\rho)|\,2^{\ell/2}r^{(d-1)/2}\times$$
$$\big\|\psi*\sigma_{r\rho}*[\chi_{R_{\mathfrak{z},\ell}} g_{z,r}](\cdot,t-br)\big\|_{L^2(\mathbb{R}^d)}\,d\rho\,\chi_{I_{z_{d+1},\ell}}(t-br)\,dt\,dr$$
$$\lesssim \sum_z\int\int_t\int_{\rho=1/2}^{2}|\omega_n(\rho)|d\rho\,|F(z,r)|\times$$
$$2^{\ell/2}r^{d-1}\big\|[\chi_{R_{\mathfrak{z},\ell}} g_{z,r}](\cdot,t-br)\big\|_{L^2(\mathbb{R}^d)}\,\chi_{I_{z_{d+1},\ell}}(t-br)\,dt\,dr.$$

Now we use (3.4), apply the Cauchy-Schwarz inequality in t and then use the normalizing assumption (3.3) to bound the last expression by

$$2^{\ell/2}\sum_z\int|F(z,r)|r^{d-1}\,2^{\ell/2}\times$$
$$\Big(\int\big\|[\chi_{R_{\mathfrak{z},\ell}} g_{z,r}](\cdot,t-br)\big\|^2_{L^2(\mathbb{R}^d)}\chi_{I_{z_{d+1},\ell}}(t-br)\,dt\Big)^{1/2}dr$$
$$\lesssim 2^\ell\sum_z\int|F(z,r)|r^{d-1}dr\,.$$

This gives the assertion for $p=\nu=1$, when $\alpha=\frac{d-1}{2}$. $\square$

4. The main estimate

In this section χ_1 will be a C^∞ function supported in $(5/8, 5/2)$ and χ will be a C^∞ function supported in $(-4,-1/4)\cup(1/4,4)$. We now consider the convolution operator T on $\mathbb{R}^{d+1}$ with multiplier

$$m(\xi,\tau)=\sum_{k\in\mathbb{Z}}\chi_1(2^{-k}|\xi|)\chi(2^{-k}\tau)\Gamma_k\big(\frac{|\xi|-b_k\tau}{2^k}\big) \tag{4.1}$$

where Γ_k is supported in $(-4,4)$, and $b_k\in(-2,2)$.

We formulate our main estimate which will, for suitable choices of Γ_k, χ, χ_1 and b_k, yield both the implications (iv) $\implies$ (i) and (iv) $\implies$ (ii) of Theorem 1.1.

Theorem 4.1. *Suppose that $1<p_1<\frac{d}{d+1}$ and that Hypothesis Sph(p_1,d) holds. Let m be as in (4.1), with $b_k\in\mathbb{R}$ and $|b_k|\le 2$. Let $1<p<p_1$ and $p\le\nu\le\infty$ and assume that*

$$\mathfrak{C}_{p,\nu}:=\sup_k\Big\|\frac{\widehat{\Gamma}_k}{(1+|\cdot|)^{\frac{d-1}{2}}}\Big\|_{L^{p,\nu}(\mathbb{R},(1+|\cdot|)^{d-1}dr)}<\infty\,. \tag{4.2}$$

Then

$$\big\|\mathcal{F}^{-1}[m\widehat{f}]\big\|_{L^{p,\nu}(\mathbb{R}^{d+1})}\lesssim\mathfrak{C}_{p,\nu}\|f\|_{L^p(\mathbb{R}^{d+1})}\,. \tag{4.3}$$

We apply the Fourier inversion formula on the real line to Γ_k and get

$$m(\xi,\tau)=\sum_k\chi(2^{-k}\tau)\chi_1(2^{-k}|\xi|)\frac{1}{2\pi}\int\widehat{\Gamma}_k(s)e^{i2^{-k}(|\xi|-b_k\tau)s}ds\,.$$

By standard singular integral theory the convolution operator with Fourier multiplier

$$\sum_k\chi(2^{-k}\tau)\chi_1(2^{-k}|\xi|)\frac{1}{2\pi}\int_{-2}^{2}\widehat{\Gamma}_k(s)e^{i2^{-k}(|\xi|-b_k\tau)s}\,ds$$

is bounded on $L^p(\mathbb{R}^{d+1})$ for all $p\in(1,\infty)$. Therefore it suffices to consider the Fourier multiplier

$$\sum_k\chi(2^{-k}\tau)\chi_1(2^{-k}|\xi|)\int_2^\infty\widehat{\Gamma}_k(s)\exp(is2^{-k}(|\xi|-\tau))ds \tag{4.4}$$

and a similar multiplier involving an integration over $(-\infty,-2)$.

We note that these multipliers define bounded functions. Their $L^\infty(\mathbb{R}^{d+1})$ norms are bounded by

$$\sup_k\int|\widehat{\Gamma}_k(s)|\,ds\lesssim\sup_k\Big\|\frac{\widehat{\Gamma}_k}{(1+|\cdot|)^{\frac{d-1}{2}}}\Big\|_{L^{p,\infty}(\nu_d)},\qquad p<\frac{2d}{d+1}; \tag{4.5}$$

here ν_d denotes the measure

$$d\nu_d(s) = (1+|s|)^{d-1} ds.$$

To see (4.5) note that the function $s \mapsto (1+|s|)^{-\frac{d-1}{2}}$ belongs to $L^{q,1}(\nu_d)$ if $q > \frac{2d}{d-1}$. Thus, for $p < \frac{2d}{d+1}$, we have

$$\int |w(s)| ds = \int \frac{|w(s)|}{(1+|s|)^{\frac{d-1}{2}}} \frac{1}{(1+|s|)^{\frac{d-1}{2}}} d\nu_d(s) \lesssim \Big\| \frac{w}{(1+|\cdot|)^{\frac{d-1}{2}}} \Big\|_{L^{p,\infty}(\nu_d)},$$

which implies (4.5).

Now let ϑ be a C^∞-function on the real line supported in $(1/8, 8)$ so that $\vartheta(s) = 1$ on $(1/7, 7)$ and observe that multiplication with $\vartheta(\beta|\xi|)$ does not affect $\chi_1(|\xi|)$ as long as $1/2 \le \beta \le 2$. Thus we have to prove that the convolution operator with multiplier

$$\sum_k \chi(\tfrac{\tau}{2^k}) \chi_1(\tfrac{|\xi|}{2^k}) \sum_{n=1}^{\infty} \int_{2^n}^{2^{n+1}} \vartheta(2^{-n} s \tfrac{|\xi|}{2^k})) \widehat{\Gamma}_k(s) \exp(is \tfrac{|\xi| - b_k \tau}{2^k})\, ds \tag{4.6}$$

is bounded on $L^p(\mathbb{R}^{d+1})$.

One can express $\mathcal{F}_d^{-1}[\exp(\pm i|\cdot|)\vartheta(2^{-n}|\cdot|)]$ as an integral over spherical means plus an error term:

Lemma 4.2. *For $n \ge 1$,*

$$\mathcal{F}_d^{-1}[e^{\pm i|\cdot|}\vartheta(2^{-n}|\cdot|)] = 2^{n(d-1)/2} \int_{1/2}^{2} \omega_n^{\pm}(\rho) \sigma_\rho d\rho \,+\, \widetilde{E}_n^{\pm} \tag{4.7}$$

where $\omega_n^{\pm}$ is smooth on $(1/2, 2)$

$$\sup_n \int |\omega_n^{\pm}(\rho)| d\rho < \infty, \tag{4.8}$$

and, for any N,

$$|\widetilde{E}_n^{\pm}(x)| + 2^{-n} |\nabla \widetilde{E}_n^{\pm}(x)| \le C_N 2^{-nN} (1+|x|)^{-N}. \tag{4.9}$$

This can be proven by an application of the stationary phase method; a more direct argument is given in Lemma 10.2 in [9].

From the lemma we see that the convolution operator with multiplier (4.6) can be split as

$$\sum_k \mathcal{K}_k * f + \sum_{n=1}^{\infty} \sum_{k\in\mathbb{Z}} 2^{k(d+1)} E_{n,k}(2^k \cdot) * f$$

where the main term is obtained by substituting the first term in (4.7) for $\mathcal{F}_d^{-1}[e^{\pm i|\cdot|}\vartheta(2^{-n}|\cdot|)]$ (*cf.* (4.11) below) and thus the rescaled term $E_{n,k}$ is given by

$$(4.10)\quad E_{n,k}(x,t)= \int_{2^n}^{2^{n+1}} \widehat{\Gamma}_k(s)\int \zeta(x-y,t-b_ks)s^{-d}\widetilde{E}_n(s^{-1}y)\,dy\,ds$$

where $\widehat{\zeta}(\xi,\tau)=\chi(\tau)\chi_1(|\xi|)$.

From (4.9) one gets

$$|E_{n,k}(x,t)|+|\nabla E_{n,k}(x,t)| \le C_{N,j}2^{-nN}\times \int_{s=2^n}^{2^{n+1}} |\widehat{\Gamma}_k(s)|\int (1+|x-y|+|t-b_ks|)^{-d-2}(1+2^{-n}|y|)^{-N/2}dy\,ds.$$

We use $(1+|x-y|+|t-b_ks|)^{-d-2}\lesssim (\frac{1+|y|+|b_ks|}{1+|x|+|t|})^{d+2}$ and since $|b_k|\le 2$ this implies (assuming N is chosen sufficiently large, say $N>10d$)

$$|E_{n,k}(x,t)|+|\nabla E_{n,k}(x,t)|\lesssim \|\widehat{\Gamma}_k\|_{L^1(\mathbb{R})}\,2^{-n}(1+|x|+|t|)^{-d-2}.$$

From this estimate it follows easily that the operator $\mathcal{E}_n$ defined by

$$\mathcal{E}_nf=\sum_{k\in\mathbb{Z}} 2^{k(d+1)}E_{n,k}(2^k\cdot)*f$$

is a Calderón-Zygmund operator which is bounded on $L^p(\mathbb{R}^{d+1})$ and the sum of the operator norms $\sum_{n=1}^\infty \|\mathcal{E}_n\|_{L^p\to L^p}$ is bounded by a constant only depending on p.

We now consider the main term. This is the operator of convolution on $\mathbb{R}^{d+1}$ with the kernel $\sum_k \mathcal{K}_k$ where

$$(4.11)\quad \mathcal{F}_{d+1}[\mathcal{K}_k](\xi,\tau)=\chi_1(2^{-k}|\xi|)\chi(2^{-k}\tau)\times \sum_{n=1}^\infty 2^{n\frac{d-1}{2}}\int_{2^n}^{2^{n+1}}\widehat{\Gamma}_k(s)e^{-ib_ks2^{-k}\tau}\int_{1/2}^2\omega_n(\rho)\mathcal{F}_d[\sigma_\rho](2^{-k}s\xi)\,d\rho\,ds\,.$$

We now let $\psi_\circ$, ψ be C_0^∞-functions as defined in the introduction and define $\eta_\circ\in\mathcal{S}(\mathbb{R}^{d+1})$ by

$$\widehat{\eta}_\circ(\xi,\tau)=\frac{\chi_1(|\xi|)\chi(\tau)}{(\widehat{\psi}_\circ(\xi))^4}=\frac{\chi_1(|\xi|)\chi(\tau)}{(\widehat{\psi}(\xi))^2}\,.$$

Define the dyadic Littlewood-Paley operator L_k by

$$\mathcal{F}_{d+1}[L_kf](\xi,\tau)=\widehat{\eta}_\circ(2^{-k}\xi,2^{-k}\tau)\mathcal{F}_{d+1}[f](\xi,\tau)\,.$$

Then

$$\mathcal{K}_k*f(x,t)=\int_2^\infty\int 2^{kd}H_{k,s}(2^k(x-y))P_kL_kf(y,t-2^{-k}b_ks)\,dy\,ds$$

where

$$\mathcal{F}_d[P_k g](\xi) = \widehat{\psi}(2^{-k}\xi)\mathcal{F}_d g(\xi) \tag{4.12}$$

and

$$\begin{aligned} H_{k,s}(x) &= \sum_{n=1}^{\infty} 2^{n\frac{d-1}{2}} \widehat{\Gamma}_k(s)\chi_{[2^n,2^{n+1})}(s) \int_{\rho=1/2}^{2} \omega_n(\rho)[\psi * s^{-d}\sigma_\rho(s^{-1}\cdot)]d\rho \\ &= \sum_{n=1}^{\infty} \widehat{\Gamma}_k(s)\chi_{[2^n,2^{n+1})}(s) 2^{n\frac{d-1}{2}} s^{1-d} \int_{\rho=1/2}^{2} \omega_n(\rho)[\psi * \sigma_{\rho s}]d\rho\,; \end{aligned} \tag{4.13}$$

in (4.13) the $*$ is used for convolution in $\mathbb{R}^d$.

Atomic decompositions. As in [9] we use atomic decompositions constructed from a nontangential Peetre type maximal square function (*cf.* [14], [18] and [16]),

$$\mathfrak{S}f(x,t) = \Big(\sum_k \sup_{|(y,s)|\le 100(d+1)\cdot 2^{-k}} |L_k f(x+y,t+s)|^2\Big)^{1/2}.$$

Then $\|\mathfrak{S}f\|_p \le C_p\|f\|_p$ for $1<p<\infty$.

For fixed k, we tile $\mathbb{R}^{d+1}$ by the dyadic cubes of sidelength 2^{-k}. We write $L(Q) = -k$ if we want to indicate that the sidelength of a dyadic cube is 2^{-k}. For each integer j, we introduce the set

$$\Omega_j = \{(x,t) : \mathfrak{S}f(x,t) > 2^j\}.$$

Let $\mathcal{Q}_j^k$ be the set of all dyadic cubes of sidelength 2^{-k} which have the property that $|Q\cap\Omega_j| \ge |Q|/2$ but $|Q\cap\Omega_{j+1}| < |Q|/2$. We also set

$$\Omega_j^* = \{(x,t) : M\chi_{\Omega_j}(x,t) > 100^{-d-1}\}$$

where M is the Hardy-Littlewood maximal operator. Ω_j^* is an open set containing Ω_j and $|\Omega_j^*| \lesssim |\Omega_j|$.

Let $\mathcal{W}_j$ is the set of all dyadic cubes W for which the 20-fold dilate of W is contained in Ω_j^* and W is maximal with respect to this property. Clearly the interiors of these cubes are disjoint and we shall refer to them as Whitney cubes for Ω_j^*. For such a Whitney cube $W\in\mathcal{W}_j$ we denote by W^* the tenfold dilate of W, and observe that the family of dilates $\{W^* : W\in\mathcal{W}_j\}$ have bounded overlap.

Note that each $Q\in\mathcal{Q}_j^k$ is contained in a unique $W\in\mathcal{W}_j$. For each $W\in\mathcal{W}_j$, set

$$A_{k,W,j} = \sum_{\substack{Q\in\mathcal{Q}_j^k\\ Q\subset W}} (L_k f)\chi_Q;$$

note that only terms with $L(W)+k\ge 0$ occur. Since any dyadic cube W can be a Whitney cube for several Ω_j^* we also define "cumulative atoms",

$$A_{k,W}=\sum_{j:W\in\mathcal{W}_j}A_{k,W,j}.$$

Standard facts about these atoms are summarized in

Lemma 4.3. *For each $j\in\mathbb{Z}$ the following inequalities hold.*

(i)

$$\sum_{W\in\mathcal{W}_j}\sum_k\|A_{k,W,j}\|_2^2\lesssim 2^{2j}\mathrm{meas}(\Omega_j).$$

(ii) There is a constant C_d such that for every assignment $W\mapsto k(W)\in\mathbb{Z}$, defined for $W\in\mathcal{W}_j$, and for $0\le p\le 2$,

$$\sum_{W\in\mathcal{W}_j}\mathrm{meas}(W)\|A_{k(W),W,j}\|_\infty^p\le C_d 2^{pj}\mathrm{meas}(\Omega_j).$$

For the proof see Lemma 7.1 in [9] (or related statements in [1], [16]).

With this notation it is now our task to show the inequality

$$\Big\|\sum_k\sum_j\sum_{\ell\ge0}\sum_{\substack{W\in\mathcal{W}_j\\ L(W)=-k+\ell}}\mathcal{K}_k*A_{k,W,j}\Big\|_{L^{p,\nu}}\lesssim\mathfrak{C}_{p,\nu}\|\mathfrak{S}f\|_p\,. \tag{4.14}$$

Let

$$\mathfrak{V}_{k,s,W,j}(x,t)\,:=\int 2^{kd}H_{k,s}(2^k(x-y))A_{k,W,j}(y,t-2^{-k}b_ks)\,dy\,, \tag{4.15}$$

with $H_{k,s}$ in (4.13) and note that

$$\mathcal{K}_k*A_{k,W,j}=P_k\Big[\int_2^\infty\mathfrak{V}_{k,s,W,j}\,ds\Big],$$

with P_k in (4.12).

The estimate (4.14) follows then from a short range and a long range inequality. The short range inequality is

$$\Big\|\sum_k\sum_j\sum_{\ell\ge0}\sum_{\substack{W\in\mathcal{W}_j\\ L(W)=-k+\ell}}P_k\big[\int_2^{2^\ell}\mathfrak{V}_{k,s,W,j}\,ds\,\big]\Big\|_{L^p} \tag{4.16}$$
$$\lesssim\sup_k\big\|\widehat{\Gamma}_k\big\|_{L^1(\mathbb{R})}\,\|\mathfrak{S}f\|_p,\qquad 1<p<2,$$

and implies the analogous $L^p \to L^{p,\nu}$ estimate since by assumption $\nu \ge p$. Recall that $\sup_k \|\widehat{\Gamma}_k\|_{L^1(\mathbb{R})} \lesssim \mathfrak{C}_{p,\infty} \lesssim \mathfrak{C}_{p,\nu}$ for $p < \frac{2d}{d+1}$, *cf.* (4.5).

The long range inequality is

$$\Big\| \sum_k \sum_j \sum_{\ell \ge 0} \sum_{\substack{W \in \mathcal{W}_j \\ L(W) = -k+\ell}} P_k \Big[\int_{2^\ell}^\infty \mathfrak{V}_{k,s,W,j}\, ds \Big] \Big\|_{L^{p,\nu}} \lesssim \mathfrak{C}_{p,\nu} \|\mathfrak{S} f\|_p . \tag{4.17}$$

The short range estimate. Since $\sum_j 2^{jp} \mathrm{meas}(\Omega_j) \lesssim \|\mathfrak{S} f\|_p^p$ it suffices by Lemma 2.1 to show that for fixed j and $1 < p < 2$

$$\begin{aligned} \Big\| \sum_k \sum_{\ell \ge 0} \sum_{\substack{W \in \mathcal{W}_j \\ L(W) = -k+\ell}} P_k \Big[\int_2^{2^\ell} \mathfrak{V}_{k,s,W,j}\, ds \Big] \Big\|_{L^p}^p \\ \lesssim \sup_k \|\widehat{\Gamma}_k\|_{L^1(\mathbb{R})}^p \, 2^{jp} \mathrm{meas}(\Omega_j). \end{aligned} \tag{4.18}$$

Here we estimate an expression which is supported in Ω_j^*. Thus the left hand side of (4.18) is dominated by

$$\mathrm{meas}(\Omega_j^*)^{1-p/2} \Big\| \sum_k \sum_{\ell \ge 0} \sum_{\substack{W \in \mathcal{W}_j \\ L(W) = -k+\ell}} P_k \Big[\int_2^{2^\ell} \mathfrak{V}_{k,s,W,j}\, ds \Big] \Big\|_{L^2}^p \tag{4.19}$$

which by the almost orthogonality of the operators P_k is dominated by a constant times

$$\mathrm{meas}(\Omega_j^*)^{1-p/2} \Big(\sum_k \Big\| \sum_{\ell \ge 0} \sum_{\substack{W \in \mathcal{W}_j \\ L(W) = -k+\ell}} \int_2^{2^\ell} \mathfrak{V}_{k,s,W,j}\, ds \Big\|_{L^2}^2 \Big)^{p/2} . \tag{4.20}$$

Now, for fixed W with $L(W) = -k + \ell$, and for every $s \le 2^\ell$, the expression $\mathfrak{V}_{k,s,W,j}$ is supported in the expanded cube W^*. The cubes W^* with $W \in \Omega_j$ have bounded overlap, and therefore the expression (4.20) is dominated by a constant times

$$\mathrm{meas}(\Omega_j^*)^{1-p/2} \Big(\sum_k \sum_{\ell \ge 0} \sum_{\substack{W \in \mathcal{W}_j \\ L(W) = -k+\ell}} \Big\| \int_2^{2^\ell} \mathfrak{V}_{k,s,W,j}\, ds \Big\|_{L^2}^2 \Big)^{p/2} . \tag{4.21}$$

Now we have for fixed W

$$\begin{aligned}
&\Big\|\int_2^{2^\ell}\mathfrak{V}_{k,s,W,j}\,ds\Big\|_{L^2(\mathbb{R}^{d+1})}\\
&\lesssim \int_2^{2^\ell}\Big(\int \big\|2^{kd}H_{k,s}(2^k\cdot)*A_{k,W,j}(\cdot,t-2^{-k}b_ks)\big\|^2_{L^2(\mathbb{R}^d)}dt\Big)^{1/2}ds\\
&\lesssim \int_2^{2^\ell}\Big(\int \big\|\mathcal{F}_d[H_{k,s}]\big\|^2_{\infty}\big\|A_{k,W,j}(\cdot,t-2^{-k}b_ks)\big\|^2_{L^2(\mathbb{R}^d)}dt\Big)^{1/2}ds\\
&=\int_2^{2^\ell}\big\|\mathcal{F}_d[H_{k,s}]\big\|_\infty\,ds\,\|A_{k,W,j}\|_{L^2(\mathbb{R}^{d+1})}
\end{aligned}$$

and

$$\begin{aligned}
&\int_2^{2^\ell}\big\|\mathcal{F}_d[H_{k,s}]\big\|_\infty ds\lesssim\\
&\qquad\sum_{n=1}^{\ell-1}\int_{2^n}^{2^{n+1}}|\widehat{\Gamma}_k(s)|2^{n\frac{d-1}{2}}s^{1-d}\int_{\rho=1/2}^2\omega_n(\rho)\big\|\mathcal{F}_d[\psi*\sigma_{\rho s}]\big\|_\infty\,d\rho\,ds\,.
\end{aligned}$$

Since $\mathcal{F}_d[\psi*\sigma_{\rho s}](\xi)=O(2^{n(d-1)/2})$ uniformly in $\rho\in(1/2,2)$ and $s\in[2^n,2^{n+1}]$. Since $\sup_n\|\omega_n\|_1\le 1$ we get

$$\int_2^{2^\ell}\big\|\mathcal{F}_d[H_{k,s}]\big\|_\infty ds\lesssim\int|\widehat{\Gamma}_k(s)|ds$$

and thus

$$\Big\|\int_2^{2^\ell}\mathfrak{V}_{k,s,W,j}\,ds\Big\|_{L^2}\lesssim\int|\widehat{\Gamma}_k(s)|ds\,\|A_{k,W,j}\|_2\,. \tag{4.22}$$

We use this estimate in (4.21). By Lemma (4.3) we have

$$\sum_k\sum_{\ell\ge0}\sum_{\substack{W\in\mathcal{W}_j\\L(W)=-k+\ell}}\big\|A_{k,W,j}\big\|_2^2\lesssim\sum_{W\in\mathcal{W}_j}\sum_k\big\|A_{k,W,j}\big\|_2^2\lesssim 2^{2j}\mathrm{meas}(\Omega_j).$$

We combine this with (4.22). Since $\mathrm{meas}(\Omega_j^*)\lesssim\mathrm{meas}(\Omega_j)$ it follows that the right hand side of (4.21) is dominated by a constant times

$$\Big[\sup_k\int|\widehat{\Gamma}_k(s)|\,ds\Big]^p\mathrm{meas}(\Omega_j)2^{jp}$$

which then yields (4.18) and finishes the proof of the short range estimate.

The long range estimate. It is now advantageous to use the cumulative atoms $A_{k,W}$. If we let

$$\mathfrak{V}_{k,s,W}(x,t):=\int 2^{kd}H_{k,s}(2^k(\cdot-y))A_{k,W}(y,t-2^{-k}b_ks)\,dy \tag{4.23}$$

then $\mathfrak{V}_{k,s,W} = \sum_j \mathfrak{V}_{k,s,W,j}$ and we have to show

(4.24)
$$\Big\|\sum_k\sum_{\ell\ge 0}\sum_{\substack{W:\\ L(W)=-k+\ell}} P_k\Big[\int_{2^\ell}^\infty \mathfrak{V}_{k,s,W}ds\Big]\Big\|_{L^{p,\nu}}^p \lesssim \mathfrak{C}_{p,\nu}^p\sum_j \mathrm{meas}(\Omega_j)2^{jp}\,.$$

By Minkowski's inequality this follows from estimates for fixed $\ell\ge 0$, with exponential decay:

$$\Big\|\sum_k\sum_{\substack{W:\\ L(W)=-k+\ell}} P_k\Big[\int_{2^\ell}^\infty \mathfrak{V}_{k,s,W}ds\Big]\Big\|_{L^{p,\nu}} \lesssim \mathfrak{C}_{p,\nu}2^{-\ell\alpha(p)}\Big(\sum_j \mathrm{meas}(\Omega_j)2^{jp}\Big)^{1/p}. \tag{4.25}$$

Here $\alpha(p)>0$ for $p<p_1$ (in fact α will be as in Proposition 3.1).

We interpolate an $L^1(\ell^1)\to L^1$ inequality and an $L^2(\ell^2)\to L^2$ inequality for the operators P_k. Let $\mathfrak{m}$ denote the measure on $\mathbb{R}^{d+1}\times\mathbb{Z}$ defined as the product measure of Lebesgue measure on $\mathbb{R}^{d+1}$ and counting measure on $\mathbb{Z}$. Define for (suitable) functions h on $\mathbb{R}^{d+1}\times\mathbb{Z}$ an operator P by $Ph(x,t)=\sum_k \mathcal{F}_d^{-1}[\widehat{\psi}(2^{-k}\cdot)\mathcal{F}_d h(\cdot,t,k)](x)$. Then P maps $L^1(\mathbb{R}^{d+1}\times\mathbb{Z},\mathfrak{m})$ to $L^1(\mathbb{R}^{d+1})$ and by orthogonality $L^2(\mathbb{R}^{d+1}\times\mathbb{Z},\mathfrak{m})$ to $L^2(\mathbb{R}^{d+1})$; thus for $1<p<2$
$$\|Ph\|_{L^{p,\nu}(\mathbb{R}^{d+1})}\lesssim \|h\|_{L^{p,\nu}(\mathbb{R}^{d+1}\times\mathbb{Z},\mathfrak{m})}\,.$$
Now by Lemma 2.2 this also implies, under the additional restriction $\nu\ge p$,
$$\Big\|\sum_k P_k f_k\Big\|_{L^{p,\nu}(\mathbb{R}^{d+1})}\lesssim\Big(\sum_k\|f_k\|_{L^{p,\nu}(\mathbb{R}^{d+1})}^p\Big)^{1/p}.$$
Using this inequality we see that (4.25) follows from

$$\sum_k\Big\|\sum_{\substack{W:\\ L(W)=-k+\ell}}\int_{2^\ell}^\infty \mathfrak{V}_{k,s,W}ds\Big\|_{L^{p,\nu}}^p\lesssim \mathfrak{C}_{p,\nu}^p 2^{-\ell p\alpha(p)}\sum_j \mathrm{meas}(\Omega_j)2^{jp}\,. \tag{4.26}$$

We need to rewrite $\int_{2^\ell}^\infty \mathfrak{V}_{k,s,W}\,ds$ and also scale it in order to apply Hypothesis $\mathrm{Sph}(p_1,d)$ (or rather its consequence stated as Proposition 3.1). Note that
$$\mathfrak{V}_{k,s,W}(x,t)=\int H_{k,s}(2^kx-y)A_{k,W}(2^{-k}y,2^{-k}(2^kt-b_ks))\,dy\,.$$
If we set
$$a_{k,W}(y,u)=A_{k,W}(2^{-k}y,2^{-k}u)$$
and
$$\mathfrak{v}_{k,s,W}(x,t)=\int H_{k,s}(x-y)a_{k,W}(y,t-b_ks)\,dy\,ds$$

then $\mathfrak{V}_{k,s,W}(x,t)=\mathfrak{v}_{k,s,W}(2^kx,2^kt)$ and of course we have

(4.27)
$$\Big\|\sum_{\substack{W:\\ L(W)=-k+\ell}}\int_{2^\ell}^\infty \mathfrak{V}_{k,s,W}ds\Big\|_{L^{p,\nu}}=2^{-k(d+1)/p}\Big\|\sum_{\substack{W:\\ L(W)=-k+\ell}}\int_{2^\ell}^\infty \mathfrak{v}_{k,s,W}ds\Big\|_{L^{p,\nu}}.$$

Next (with $*$ denoting convolution in $\mathbb{R}^d$)

$$\int_{2^\ell}^\infty \mathfrak{v}_{\kappa,r,W}(x,t)dr= \tag{4.28}$$
$$\int_{1/2}^2\sum_{n=\ell}^\infty\omega_n(\rho)\int_{2^n}^{2^{n+1}}\widehat{\Gamma}_k(r)r^{1-d}2^{n(d-1)/2}[\psi*\sigma_{\rho r}*a_{k,W}](x,t-b_kr)\,dr\,d\rho\,.$$

We are now in the position to apply Proposition 3.1, with the choice of

$$F(z,r)\equiv F_{k,\ell}(z,r)\ =$$
$$\sum_{n=\ell}^\infty\widehat{\Gamma}_k(r)r^{1-d}2^{n\frac{d-1}{2}}\chi_{[2^n,2^{n+1}]}(r)\Big\|\sum_{W:L(W)=\ell-k}A_{k,W}(2^{-k}\cdot)\chi_{z,\ell}\Big\|_2\,.$$

The sum in W collapses as for given $z=(\mathfrak{z},z_{d+1})$ there is a unique dyadic cube W of sidelength $2^{\ell-k}$ so that the dyadic cube $2^kW=\{2^k(x,t):(x,t)\in W\}$ is equal to $R_{\mathfrak{z},\ell}\times I_{z_{d+1},\ell}$. Also observe the pointwise estimate

$$\sum_{n=\ell}^\infty|\widehat{\Gamma}_k(r)|r^{1-d}2^{n\frac{d-1}{2}}\chi_{[2^n,2^{n+1}]}(r)\lesssim\frac{|\widehat{\Gamma}_k(r)|}{(1+|r|)^{\frac{d-1}{2}}}.$$

We now proceed to finish the proof of (4.26). By Proposition 3.1 and the Fubini-type Lemma 2.2 we get from (4.28)

$$\Big\|\sum_{\substack{W:\\ L(W)=-k+\ell}}\int_{2^\ell}^\infty\mathfrak{v}_{k,s,W}ds\Big\|_{L^{p,\nu}}\lesssim 2^{\ell((d+1)(\frac1p-\frac12)-\alpha)}\|F_{k,\ell}\|_{L^{p,\nu}(\mathbb{Z}^{d+1}\times\mathbb{R}^+,\mu_d)}$$

(4.29)
$$\lesssim 2^{\ell((d+1)(\frac1p-\frac12)-\alpha)}\Big\|\frac{|\widehat{\Gamma}_k|}{(1+|\cdot|)^{\frac{d-1}{2}}}\Big\|_{L^{p,\nu}(\mathbb{R},(1+|r|)^{d-1}dr)}\quad\times$$
$$\Big(\sum_{\substack{W:\\ L(W)=-k+\ell}}\big\|A_{k,W}(2^{-k}\cdot)\big\|_2^p\Big)^{\frac1p}$$

where α is as in Proposition 3.1. Combining (4.27) and (4.29) we obtain after a change of variables

$$\Big\| \sum_{\substack{W:\\ L(W)=-k+\ell}} \int_{2^\ell}^\infty \mathfrak{V}_{k,s,W} ds \Big\|_{L^{p,\nu}} \lesssim \mathfrak{C}_{p,\nu} 2^{-\ell\alpha} 2^{(\ell-k)(d+1)(\frac1p-\frac12)} \Big(\sum_{\substack{W:\\ L(W)=-k+\ell}} \big\| A_{k,W} \big\|_2^p \Big)^{1/p}. \tag{4.30}$$

Note that for fixed k and W, the functions $A_{k,W,j}$ live on disjoint sets (since the dyadic cubes of sidelength 2^{-k} are disjoint and each such cube is in exactly one family $\mathcal{Q}_j^k$). Thus $\|A_{k,W}\|_2^p \lesssim \sum_j \|A_{k,W,j}\|_2^p$. We now sum in k and obtain from (4.30)

$$\Big(\sum_k \Big\| \sum_{\substack{W:\\ L(W)=-k+\ell}} \int_{2^\ell}^\infty \mathfrak{V}_{k,s,W} ds \Big\|_{L^{p,\nu}}^p \Big)^{1/p} \lesssim \mathfrak{C}_{p,\nu} 2^{-\ell\alpha} \Big(\sum_k \sum_j \sum_{\substack{W\in\mathcal{W}_j:\\ L(W)=-k+\ell}} \operatorname{meas}(W)^{1-p/2} \big\| A_{k,W,j} \big\|_2^p \Big)^{1/p}.$$

Finally, using part (ii) of Lemma 4.3, we get

$$\begin{aligned} &\Big(\sum_k \sum_j \sum_{\substack{W\in\mathcal{W}_j:\\ L(W)=-k+\ell}} \operatorname{meas}(W)^{1-p/2} \big\| A_{k,W,j} \big\|_2^p \Big)^{1/p} \\ &\lesssim \Big(\sum_j \sum_{W\in\mathcal{W}_j} \operatorname{meas}(W) \big\| A_{\ell-L(W),W,j} \big\|_\infty^p \Big)^{1/p} \\ &\lesssim \Big(\sum_j \operatorname{meas}(\Omega_j)\, 2^{jp} \Big)^{1/p} \lesssim \|\mathfrak{S} f\|_p . \end{aligned}$$

This finishes the proof of (4.26).

□

5. Proof of Theorem 1.1

By the remarks in the introduction (following the statement of Corollary 1.2) it only remains to be shown that (iv) implies (i) and (ii). These implications quickly follow from Theorem 4.1.

For the implication (iv) $\Longrightarrow$ (i) we show, for the choices $b = 1$ and $b = \sqrt{2}$ that the multiplier

$$m(\xi,\tau) = \sum_k \mathbb{1}_{[2^k, 2^k\sqrt{2})}(\tau)\gamma_k\big(2^{-k}(|\xi| - b\tau)\big) \tag{5.1}$$

defines an operator which is bounded from $L^p(\mathbb{R}^{d+1})$ to $L^{p,\nu}(\mathbb{R}^{d+1})$. The choice $b = \sqrt{2}$ and scaling in τ then also covers the multiplier

$$m(\xi,\tau) = \sum_k \mathbb{1}_{[2^k\sqrt{2}, 2^{k+1})}(\tau)\gamma_k\big(2^{-k}(|\xi| - \tau)\big)$$

and the assertion follows.

For the proof of (5.1) pick a smooth function χ_2 which is equal to one on $[1, \sqrt{2}]$ and supported in $(9/10, 3/2)$. Recall that γ_k is supported in $(-1/4, 1/4)$ and pick a smooth function χ_1 which is equal to one on the interval $(\frac{9}{10}b - \frac{1}{4}, \frac{3}{2}b + \frac{1}{4})$ and supported in $(\frac{5b}{8}, 2b)$ (in particular, for $b = 1$ or $\sqrt{2}$, χ_1 is supported in $(\frac{5}{8}, \frac{5}{2})$ as in §4). Observe that, with these definitions

$$m(\xi,\tau) = \chi_E(\tau)\sum_k \chi_2(2^{-k}\tau)\chi_1(2^{-k}|\xi|)\gamma_k\big(2^{-k}(|\xi| - b\tau)\big) \tag{5.2}$$

where $E = \bigcup_{k\in\mathbb{Z}}[2^k, 2^{k+\frac{1}{2}})$. By the Marcinkiewicz multiplier theorem the convolution with multiplier $\chi_E(\tau)$ is bounded on $L^{p,\nu}(\mathbb{R}^{d+1})$ for all $1 < p < \infty$, $0 < \nu \le \infty$. Therefore it suffices to prove that under condition (1.4) the multiplier

$$\sum_k \gamma_k\big(2^{-k}(|\xi| - b\tau)\big)\chi_2(2^{-k}\tau)\chi_1(2^{-k}|\xi|) \tag{5.3}$$

defines a convolution which maps $L^p(\mathbb{R}^{d+1})$ to $L^{p,\nu}(\mathbb{R}^{d+1})$. But this follows immediately from Theorem 4.1, with the choice of $\Gamma_k = \gamma_k$, and $b_k = b$ ($= 1$ or $\sqrt{2}$) for all $k \in \mathbb{Z}$.

Next, for the implication (iv) $\Longrightarrow$ (ii) we first note that since $\tau_k \in [2^k, 2^{k+1}]$ the term $\gamma(2^{-k}(|\xi| - \tau_k))$ vanishes for $|\xi| \notin (\frac{3}{4}2^k, \frac{9}{4}2^k)$. Now choose χ_1 so that χ_1 is supported in $(5/8, 5/2)$ and equal to one on $(3/4, 9/4)$. Then

$$\mathcal{F}_d[\sum_k \alpha_k T^{\tau_k} f](\xi) = \sum_k \alpha_k\gamma_k(2^{-k}|\xi| - 2^{-k}\tau_k)\chi_1(2^{-k}|\xi|)\mathcal{F}_d[f](\xi)\,.$$

Now let χ be smooth and compactly supported in $(-4, 4)$. We claim that the multiplier transformation with Fourier muliplier

$$M(\xi,\tau) = \sum_k \alpha_k\gamma_k\big(2^{-k}(|\xi| - \tau_k)\big)\chi_1(2^{-k}|\xi|)\,\chi(2^{-k}\tau) \tag{5.4}$$

maps $L^p(\mathbb{R}^{d+1})$ to $L^{p,\nu}(\mathbb{R}^{d+1})$. To see this we apply Theorem 4.1 with $\Gamma_k(s) = \alpha_k\gamma_k(s - 2^{-k}\tau_k)$ and $b_k = 0$ for all $k \in \mathbb{Z}$. The condition (1.4) for $\widehat{\gamma_k}$ is obviously equivalent with the condition (4.2) for $\widehat{\Gamma}_k$.

Now in (5.4) $\chi(\tau)$ may be chosen so that $\chi(0) = 1$. With this choice it follows by de Leeuw's theorem (Lemma 2.3) that $\sum_k \alpha_k T^{\tau_k}$ maps $L^p(\mathbb{R}^d)$ to $L^{p,\nu}(\mathbb{R}^d)$. □

6. The cone multiplier

Proof of Corollary 1.3. It suffices to consider the multiplier $\rho_\lambda(\xi,\tau)\chi_{(0,\infty)}(\tau)$. We split for $\tau > 0$

$$\rho_\lambda(\xi,\tau) = \sum_{k\in\mathbb{Z}} \mathbb{1}_{[2^k,2^{k+1})}(\tau)\frac{2^{k\lambda}(\tau+|\xi|)^\lambda}{\tau^{2\lambda}}\Big(\frac{\tau-|\xi|}{2^k}\Big)_+^\lambda.$$

Now let $\beta_\circ \in C_c^\infty(\mathbb{R})$ be supported in $(-1/4, 4)$ and satisfy $\beta_\circ(s) = 1$ for $|s| \le 1/8$. We can then write

$$\rho_\lambda(\xi,\tau)\chi_{(0,\infty)}(\tau) = a_\lambda(\xi,\tau)\sum_{k\in\mathbb{Z}} \mathbb{1}_{[2^k,2^{k+1})}(\tau)\gamma\big(\frac{|\xi|-\tau}{2^k}\big) + \widetilde{a}_\lambda(\xi,\tau)$$

where

$$a_\lambda(\xi,\tau) = \sum_{k\in\mathbb{Z}} \mathbb{1}_{[2^k,2^{k+1})}(\tau)\Big(\frac{2^k(\tau+|\xi|)}{\tau^2}\Big)^\lambda \beta_\circ\big(\frac{|\xi|-\tau}{2^k}\big)$$

$$\widetilde{a}_\lambda(\xi,\tau) = \sum_{k\in\mathbb{Z}} \mathbb{1}_{[2^k,2^{k+1})}(\tau)\Big(1-\frac{|\xi|^2}{\tau^2}\Big)_+^\lambda \big(1-\beta_\circ^2\big(\frac{|\xi|-\tau}{2^k}\big)\big)$$

and

$$\gamma(u) = \begin{cases} (-u)^\lambda \beta_\circ(u) & \text{for } u < 0 \\ 0 & \text{for } u > 0 \end{cases}.$$

The multipliers a_λ and $\widetilde{a}_\lambda$ are treated by the Marcinkiewicz multiplier theorem. The associated convolution operators are thus bounded on $L^{p,\nu}$ for all $1 < p < \infty$, $0 < \nu \le \infty$. Therefore the corollary follows if we can show that the convolution operator with multiplier

$$\sum_{k\in\mathbb{Z}} \mathbb{1}_{[2^k,2^{k+1})}(\tau)\gamma(2^{-k}(|\xi|-\tau))$$

maps $L^p(\mathbb{R}^{d+1})$ boundedly to $L^{p,\infty}(\mathbb{R}^{d+1})$. By Theorem 1.1 this is the case if

$$\Big\|\frac{\widehat{\gamma}}{(1+|\cdot|)^{\frac{d-1}{2}}}\Big\|_{L^{p,\infty}(\mathbb{R},(1+|\cdot|)^{d-1}dr)} < \infty.$$

But

$$|\widehat{\gamma}(s)| \le C(1+|s|)^{-\lambda-1}$$

and it is easy to check that $(1+|\cdot|)^{-\lambda-1-\frac{d-1}{2}}$ belongs to $L^{p,\infty}(\mathbb{R},(1+|r|)^{d-1}dr)$ if and only if $\lambda \ge d/p-(d+1)/2$. □

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