

## Math 522, Spring 2016

### I. An existence and uniqueness theorem for differential equations

We are concerned with the initial value problem for a differential equation

$$(1) \quad \begin{aligned} y'(t) &= F(t, y(t)), \\ y(t_0) &= y_0. \end{aligned}$$

Here  $(t, y) \mapsto F(t, y)$  is a given continuous function of two variables defined near a point  $(t_0, y_0) \in \mathbb{R}^2$ . We wish to find a function  $t \mapsto y(t)$  for which the derivative  $y'(t)$  equals the value of  $F(t, y)$  for  $y = y(t)$ , and which has the value  $y_0$  at the “initial” time  $t_0$ .

If we impose an additional condition on  $F$  (that is, (4) below) then this problem has a unique solution in some open interval containing  $t_0$ . The following theorem also gives information about a lower bound for the length of the interval on which this solution exists.

**The Picard-Lindelöf Theorem.** *Let  $\Omega$  be a nonempty open set in  $\mathbb{R} \times \mathbb{R}$ , let  $(t_0, y_0) \in \Omega$  and let  $F : \Omega \rightarrow \mathbb{R}$  be continuous. Let  $\mathcal{R}$  be a compact rectangle of the form*

$$\mathcal{R} = \{(t, y) : |t - t_0| \leq a, \quad |y - y_0| \leq b\}$$

*contained in  $\Omega$ . Suppose that*

$$(2) \quad |F(t, y)| \leq M \text{ for } (t, y) \in \mathcal{R}$$

*and let*

$$(3) \quad a_* := \min\{a, b/M\}.$$

*In addition assume that there is a constant  $C$  so that*

$$(4) \quad |F(t, y) - F(t, u)| \leq C|y - u| \text{ whenever } (t, y) \in \mathcal{R}, (t, u) \in \mathcal{R}.$$

*Then there exists a unique function  $t \mapsto y(t)$  defined on  $[t_0 - a_*, t_0 + a_*]$  which satisfies the initial value problem*

$$\begin{aligned} y'(t) &= F(t, y(t)) \text{ for } |t - t_0| \leq a_* \\ y(t_0) &= y_0. \end{aligned}$$

*Remarks:* (i) In the theorem the expression  $y'(t)$  makes sense at the end-points of the interval if we take a one-sided derivative, i.e.  $y'(t_0 + a_*) = \lim_{h \rightarrow 0+} \frac{y(t_0 + a_* + h) - y(t_0 + a_*)}{h}$  and  $y'(t_0 - a_*) = \lim_{h \rightarrow 0+} \frac{y(t_0 - a_* + h) - y(t_0 - a_*)}{h}$ .

(ii) The hypothesis (4) says that on  $\mathcal{R}$  the function  $F$  satisfies a Lipschitz-condition with respect to the variable  $y$ . In the case where  $F$  is differentiable

with respect to  $y$  and the partial derivative  $\frac{\partial F}{\partial y}$  satisfies the bound

$$\left| \frac{\partial F}{\partial y}(t, y) \right| \leq C \text{ for } (t, y) \in \mathcal{R},$$

the hypothesis (4) is satisfied. This is a consequence of the mean value theorem applied to  $y \mapsto F(t, y)$ , for any fixed  $t$ .

**Equivalence of the initial value problem and an integral equation.** Suppose that on  $[t_0 - a_*, t_0 + a_*]$  there is a  $C^1$  function  $y$  with values in  $[y_0 - b, y_0 + b]$  so that  $y'(t) = F(t, y(t))$  and  $y(t_0) = y_0$ . Then by integrating (using the fundamental theorem of calculus) we have

$$y(t) = y_{t_0} + \int_{t_0}^t y'(s) ds$$

and thus

$$y(t) = y_0 + \int_{t_0}^t F(s, y(s)) ds$$

for  $t \in [y_0 - a_*, y_0 + a_*]$ . Vice versa, if there is a continuous function  $y$  satisfying the displayed integral equation, then  $y(t_0) = y_0$  and the integrand  $s \mapsto F(s, y(s))$  is continuous. By the fundamental theorem of calculus the right hand side defines a differentiable function. We differentiate and get  $y'(t) = F(t, y(t))$ .

**Proof of the Picard-Lindelöf Theorem.** By the previous section we need to show that there is a unique continuous function on

$$I := [t_0 - a_*, t_0 + a_*]$$

which takes values  $[y_0 - b, y_0 + b]$  so that the integral equation

$$(5) \quad y(t) = y_0 + \int_{t_0}^t F(s, y(s)) ds$$

has a unique solution.

Define  $\mathcal{Y}$  as the set of continuous functions  $t \mapsto y(t)$  which are defined on  $[t_0 - a_*, t_0 + a_*]$  and which take values in  $[y_0 - b, y_0 + b]$ .

Note that for each  $y \in \mathcal{Y}$  the function  $t \mapsto F(t, y(t))$  is well defined and a continuous function on  $[t_0 - a_*, t_0 + a_*]$ . We may therefore define for  $y \in \mathcal{Y}$  a function  $Ty$  by

$$Ty(t) = y_0 + \int_{t_0}^t F(s, y(s)) ds.$$

If  $y \in \mathcal{Y}$  then  $Ty$  is a continuous function on  $[t_0 - a_*, t_0 + a_*]$ . We *claim* that it is also in  $\mathcal{Y}$ , i.e. takes values in  $[y_0 - b, y_0 + b]$ . To see this we estimate

$$\begin{aligned} |Ty(t) - y_0| &= \left| \int_{t_0}^t F(s, y(s)) ds \right| \\ &\leq |t_0 - t| \max_{(s,u) \in \mathcal{R}} |F(s, u)| \\ &\leq M|t - t_0| \leq Ma_* \leq b \end{aligned}$$

Here we have used the definitions of  $M$  and  $a_*$  in the statement of the theorem. Thus the claim is proved.

We have thus proved that  $T$  maps  $\mathcal{Y}$  into  $\mathcal{Y}$ . It is our goal to show that with a suitable metric the map  $T : \mathcal{Y} \rightarrow \mathcal{Y}$  is a contraction, since this shows the existence of a unique fixed point of  $T$  and hence a unique solution to (5).

Now what does "suitable metric" mean? One natural choice would certainly be the usual maximum's metric  $d(g_1, g_2) = \sup_{t \in I} |g_1(t) - g_2(t)|$  and it is clear that  $\mathcal{Y}$  is a closed subspace of  $C(I)$ , with respect to this metric, hence a complete metric space. However we will prefer to work with <sup>1</sup>

$$d_C(g_1, g_2) = \sup_{x \in I} e^{-2C|x-t_0|} |g_1(x) - g_2(x)|$$

which is an equivalent metric on  $I$  in the sense that

$$d_C(g_1, g_2) \leq d(g_1, g_2) \leq e^{2Ca_*} d_C(g_1, g_2)$$

for all  $g_1, g_2$ . It is immediate that  $\mathcal{Y}$  with the metric  $d_C$  is also complete.

We now show that  $T : \mathcal{Y} \rightarrow \mathcal{Y}$  is a contraction with respect to the metric  $d_C$ , indeed we shall verify that

$$(6) \quad d_C(Ty_1, Ty_2) \leq \frac{1}{2} d_C(y_1, y_2)$$

To verify (6) we examine  $\mathcal{T}g_2(t) - \mathcal{T}g_1(t)$ . First observe that

$$\mathcal{T}g_2(t) - \mathcal{T}g_1(t) = \int_{t_0}^t [F(s, g_2(s)) - F(s, g_1(s))] ds.$$

By the Lipschitz condition (4), we have

$$|\mathcal{T}g_2(t) - \mathcal{T}g_1(t)| \leq \begin{cases} \int_{t_0}^t C|g_2(s) - g_1(s)| ds & \text{if } t \geq t_0 \\ \int_t^{t_0} C|g_2(s) - g_1(s)| ds & \text{if } t \leq t_0 \end{cases}.$$

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<sup>1</sup>If in the conclusion of the theorem we replace  $a_*$  with a smaller number  $a_{**}$  satisfying  $a_{**} < \min\{a, b/M, 1/C\}$  then it suffices to work with the standard sup-metric  $d$ .

Now we derive an estimate for  $t \in [t_0, t_0 + a_*]$  and then a similar estimate for  $t \in [t_0 - a_*, t_0]$ . For  $t_0 \leq t \leq t_0 + a_*$

$$\begin{aligned}
|\mathcal{T}g_2(t) - \mathcal{T}g_1(t)| &\leq \int_{t_0}^t C|g_2(s) - g_1(s)| ds \\
&= C \int_{t_0}^t e^{2C(s-t_0)} e^{-2C(s-t_0)} |g_2(s) - g_1(s)| ds \\
&\leq C \int_{t_0}^t e^{2C(s-t_0)} d_C(g_1, g_2) ds = C d_C(g_1, g_2) \int_{t_0}^t e^{2C(s-t_0)} ds \\
&= C d_C(g_1, g_2) \frac{e^{2C(t-t_0)} - 1}{2C} \leq \frac{1}{2} e^{2C|t-t_0|} d_C(g_1, g_2).
\end{aligned}$$

Similarly, for  $t_0 - a_* \leq t \leq t_0$ ,

$$\begin{aligned}
|\mathcal{T}g_2(t) - \mathcal{T}g_1(t)| &\leq \int_t^{t_0} C|g_2(s) - g_1(s)| ds \\
&= C \int_t^{t_0} e^{2C(t_0-t)} e^{-2C(t_0-s)} |g_2(s) - g_1(s)| ds \\
&\leq C \int_t^{t_0} e^{2C(t_0-s)} d_C(g_1, g_2) ds \\
&= C d_C(g_1, g_2) \frac{e^{2C(t_0-t)} - 1}{2C} \leq \frac{1}{2} e^{2C|t_0-t|} d_C(g_1, g_2).
\end{aligned}$$

Combining both cases we see that

$$e^{-2C|t-t_0|} |\mathcal{T}g_2(t) - \mathcal{T}g_1(t)| \leq \frac{1}{2} d_C(g_1, g_2) \text{ whenever } |t - t_0| \leq a_*.$$

Now take the sup over  $|t - t_0| \leq a_*$  on the left hand side and we get  $d_C(\mathcal{T}g_1, \mathcal{T}g_2) \leq \frac{1}{2} d_C(g_1, g_2)$  as claimed in (6). This finishes the proof.  $\square$

**The method of successive approximation.** The proof of the contraction principle tells us that for given  $Y_0 \in \mathcal{Y}$  the sequence  $Y_n$  defined recursively by  $Y_n = TY_{n-1}$  converges to the unique function satisfying  $y = Ty$  which is the solution to our integral equation (5) and thus the solution of our initial value problem. It is advisable to choose for the initial function  $Y_0$  the constant function  $y_0$  so that the iteration becomes

$$\begin{aligned}
Y_0(t) &= y_0 \\
Y_n(t) &= y_0 + \int_{t_0}^t F(s, Y_{n-1}(s)) ds, \quad n = 1, 2, \dots
\end{aligned}$$

**How to apply the theorem: An example.**

Suppose you are given the problem

$$\begin{aligned}
(1 - x^2 y(x)^2) y'(x) - e^{y(x)^2 - 1} &= 0 \\
y(-2) &= 1
\end{aligned}$$

which you likely cannot solve explicitly. We want to identify an interval centered at  $-2$  on which this problem has a unique solution. First we need to rewrite the differential equation in the form  $y'(x) = F(x, y(x))$ , and our problem is equivalent to

$$\begin{aligned} y'(x) &= \frac{e^{y(x)^2-1}}{1-x^2y(x)^2} \\ y(-2) &= 1 \end{aligned}$$

We want to find an interval on which a solution surely exists. Here our function  $F$  is defined by  $F(x, y) = e^{y^2-1}(1-x^2y^2)^{-1}$  and  $x_0, y_0$  are given by  $x_0 = -2, y_0 = 1$ . Thus we need to pick a rectangle  $\mathcal{R}$  which is centered at  $(-2, 1)$ . In this rectangle we need to have good control on  $F$  and  $\partial F/\partial y$  and so we certainly have to choose  $R$  so small that it contains no points at which the denominator  $1-x^2y^2$  vanishes. The exact choice of the rectangle is up to you but the properties of  $F$  and  $\partial F/\partial y$ , as required in the theorem, must be satisfied.

Let's pick  $a, b$  small in the definition of  $\mathcal{R}$ , say, let's choose  $a = 1/2$  and  $b = 1/4$  so that we work in the rectangle

$$\mathcal{R} = \{(x, y) : -5/2 \leq x \leq -3/2, 3/4 \leq y \leq 5/4\}.$$

Notice that then for  $(x, y)$  in  $R$  we have  $x^2 \geq 9/4, y^2 \geq 9/16$  and therefore  $x^2y^2 \geq 81/64$  so  $|1-x^2y^2| \geq \frac{81}{64} - 1 = 17/64 > 1/4$ . for  $(x, y)$  in  $R$ . Thus we get  $|(1-x^2y^2)^{-1}| < 4$  and  $e^{y^2-1} \leq e^{\frac{25}{16}-1} = e^{9/16} < 3$  which implies

$$|F(x, y)| = \left| \frac{e^{y^2-1}}{1-x^2y^2} \right| \leq 3 \cdot 4 = 12, \quad \text{for } (x, y) \text{ in } R.$$

Thus a legitimate (but non-optimal) choice for  $M$  in (1.3) is  $M = 12$ .

To verify also the Lipschitz condition we compute

$$\frac{\partial F}{\partial y}(x, y) = \frac{2ye^{y^2-1}}{1-x^2y^2} + \frac{e^{y^2-1}}{(1-x^2y^2)^2} 2yx^2.$$

Observe that that  $|2y| \leq 5/2$  and  $|2yx^2| \leq (5/2)^3$  in  $R$  and using the bounds above we can estimate for all  $(x, y)$  in  $R$

$$\begin{aligned} \left| \frac{\partial F}{\partial y}(x, y) \right| &\leq \left| \frac{2ye^{y^2-1}}{1-x^2y^2} \right| + \left| \frac{e^{y^2-1}}{(1-x^2y^2)^2} 2yx^2 \right| \\ &\leq \frac{5}{2} \cdot 12 + 3 \cdot 4^2 \cdot (5/2)^3 = 780. \end{aligned}$$

Hence the Lipschitz condition holds with constant  $C = 780$ .

Now if we take

$$h \leq \min \left\{ a, \frac{b}{M} \right\} = \min \{ 1/2, (1/4)/12 \} = 1/48,$$

then by Picard's theorem we know the problem has exactly one solution in the interval  $[-2-h, -2+h]$ . So for example if we chose  $h = .02$  we would deduce that there is a unique solution in the interval  $[-2.02, -1.98]$ .

Finally, you may be asked to implement the method of successive approximation, i.e. write down the sequence  $Y_n(x)$ . Set  $Y_0(x) = 1$ ,

$$Y_n(x) = 1 + \int_{-2}^x \frac{e^{Y_{n-1}(s)^2-1}}{1-s^2(Y_{n-1}(s))^2} ds$$

You can compute  $Y_1(x) = 1 + \int_{-2}^x \frac{1}{1-s^2} ds$ , but the explicit computation of  $Y_2$  already needs to be done numerically (write out these formulas).

**A simple exercise on successive approximation.** Consider

$$y'(t) = \gamma y(t), \quad y(t_0) = y_0.$$

Everybody has learned in calculus how to solve this problem. Now determine an explicit formula for the iterations in the method of successive approximation.

$$Y_n(t) = y_0 + \int_{t_0}^t \gamma Y_{n-1}(s) ds$$

with  $Y_0(t) = y_0$ .

One verifies that

$$Y_1(t) = y_0 + \int_{t_0}^t \gamma y_0 = y_0 + y_0 \gamma(t - t_0),$$

$$Y_2(t) = y_0 + \int_{t_0}^t \gamma(y_0 + y_0 \gamma(s - t_0)) ds = y_0 \left( 1 + \gamma(t - t_0) + \gamma^2 \frac{(t - t_0)^2}{2} \right)$$

and then proves by induction that

$$Y_n(t) = y_0 \sum_{k=0}^n \frac{\gamma^k (t - t_0)^k}{k!}.$$

Note that  $\lim_{n \rightarrow \infty} Y_n(t) = y_0 e^{\gamma(t-t_0)}$  which is indeed the solution to our initial value problem.

**Another example.** Consider the initial value problem

$$\begin{aligned} y'(x) &= 2 \sin(3xy(x)) \\ y(0) &= y_0 \end{aligned},$$

for any choice of  $y_0$ . We use Picard's theorem to show that this problem has a unique solution in  $(-\infty, \infty)$ . To do this it suffices to show that it has a unique solution on every interval  $[-L, L]$ .

This is because we have a unique solution on the interval  $[-L_1, L_1]$  and a unique solution on the interval  $[-L_2, L_2]$  with  $L_2 > L_1$  by the uniqueness part the two solutions have to agree on the smaller interval  $[-L_1, L_1]$ .

Now fix  $L$ . Define

$$\mathcal{R} = \{(x, y) : -L \leq x \leq L, y_0 - b \leq y \leq y_0 + b\}$$

for (large)  $b$ . Note that the function  $F$  defined by  $F(x, y) = 2 \sin(3xy)$  satisfies  $|F(x, y)| \leq 2$  and  $|\partial F / \partial y| \leq 6L$  for  $(x, y)$  in  $\mathcal{R}$ ; in particular observe

that these bounds are independent of  $b$ . By the existence and uniqueness theorem there is a unique solution for the problem on the interval  $[-h, h]$  where  $h = \min\{L, b/2\}$ . Since our bounds are independent of  $b$  we may choose  $b$  large, in particular we may choose  $b$  larger than  $2L$ , so that  $h = L$  is permissible. Thus we get a unique solution on  $[-L, L]$ .

## II. Possible failure of uniqueness in the absence of the Lipschitz condition

If in Picard's theorem one drops the Lipschitz condition then there may be more than one solution, thus the uniqueness assertion in the theorem is not longer valid.

We give an example. Here  $F(t, y) = \sqrt{|y|}$  which is continuous on  $\mathbb{R}^2$  but does not satisfy a Lipschitz condition in any rectangle containing  $(0, 0)$  in its interior.

Consider the initial value problem

$$\begin{aligned} y'(t) &= \sqrt{|y(t)|}, \\ y(0) &= 0. \end{aligned}$$

Verify that the four functions

$$\begin{aligned} Y_1(t) &= 0 & Y_2(t) &= \begin{cases} t^2/4 & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases} \\ Y_3(t) &= \begin{cases} 0 & \text{if } t > 0 \\ -t^2/4 & \text{if } t \leq 0 \end{cases} & Y_4(t) &= \begin{cases} t^2/4 & \text{if } t > 0 \\ -t^2/4 & \text{if } t \leq 0 \end{cases} \end{aligned}$$

are differentiable on  $\mathbb{R}$  and are solutions of the given initial value problem.

## III. A more general existence theorem

We have just seen that the uniqueness part in Picard's theorem fails to hold if one drops the Lipschitz assumption in the  $y$ -variable, (cf. (4)). However the existence part remains true. This was shown by Peano and can be seen as an application of compactness, i.e. the Arzelà-Ascoli theorem.

**Peano's existence theorem.** *Let  $\Omega$  be a nonempty open set in  $\mathbb{R} \times \mathbb{R}$ , let  $(t_0, y_0) \in \Omega$  and let  $F : \Omega \rightarrow \mathbb{R}$  be continuous. Let  $\mathcal{R}$  be a compact rectangle of the form*

$$\mathcal{R} = \{(t, y) : |t - t_0| \leq a, \quad |y - y_0| \leq b\}$$

*contained in  $\Omega$ . Suppose that  $|F(t, y)| \leq M$  for  $(t, y) \in \mathcal{R}$  and let  $a_* = \min\{a, b/M\}$ . Then there exists a function  $t \mapsto y(t)$  defined on  $[t_0 - a_*, t_0 +$*

$a_*$ ] which satisfies the initial value problem

$$\begin{aligned} y'(t) &= F(t, y(t)) \quad \text{for } |t - t_0| \leq a_* \\ y(t_0) &= y_0. \end{aligned}$$

**Proof.** <sup>2</sup>The strategy is again to prove the existence of a continuous function  $y$  on  $[t_0 - a_*, t_0 + a_*]$  which satisfies the integral equation

$$y(t) = y_0 + \int_{t_0}^t F(s, y(s)) ds.$$

Once we have found this function we observe that the integrand  $F(s, y(s))$  is also a continuous function. Thus the integral represents a differentiable function (by the fundamental theorem of calculus) and its derivative is  $F(t, y(t))$ . Thus  $y$  is also differentiable and we have  $y' = F(t, y(t))$  for  $t \in (t_0 - a_*, t_0 + a_*)$ , i.e.  $y$  is a solution of the differential equation. Also by the last display  $y(t_0) = y_0$  and a solution is found.

We write up the proof of the existence of  $y$  only for the interval  $[t_0, t_0 + a_*]$  and leave the notational changes for the interval  $[t_0 - a_*, t_0]$  to the reader. We will split the proof into five steps.

**1.** We shall construct functions with polygonal graphs which are candidates to approximate the solutions.

$F$  is uniformly continuous on the compact set  $\mathcal{R}$ . Let  $\varepsilon > 0$  a small number and let  $\delta = \delta(\varepsilon)$  be as in the definition of uniform continuity, i.e. we have

$$\begin{aligned} |F(t, y) - F(t', y')| &< \varepsilon \\ \text{whenever } |(t, y) - (t', y')| &\leq \delta \text{ and } (t, y) \in \mathcal{R}, (t', y') \in \mathcal{R}. \end{aligned}$$

Let

$$t_0 < t_1 < \cdots < t_N = t_0 + a_*$$

be a partition of  $[t_0, t_0 + a_*]$  so that  $t_{k+1} - t_k < \frac{1}{2} \min\{\delta, \delta/M\}$  for  $k = 0, \dots, N-1$ .

We now construct a function  $Y \equiv Y_\varepsilon$  on  $[t_0, a_*]$ ; this definition depends on  $\varepsilon$ ,  $\delta$  and the partition chosen, however keeping this dependence in mind we will omit the subscript  $\varepsilon$  in steps 1-4 to avoid cluttered notation.

To define  $Y \equiv Y_\varepsilon$  we set  $Y(t_0) = y_0$ . On the first partition interval  $[t_0, t_1]$  the graph of  $Y$  will be a line with initial point  $(t_0, Y(t_0))$  and slope  $F(t_0, y_0) = F(t_0, Y(t_0))$ . The value of this function at  $t_1$  is  $Y(t_1) = Y(t_0) + F(t_0, Y(t_0))(t_1 - t_0)$ . On the interval  $(t_1, t_2]$  we wish to define  $Y$  as the graph of a line starting at  $(t_1, Y(t_1))$  with readjusted slope  $F(t_1, Y(t_1))$ . In order for this construction to work we need to make sure that  $F(t_1, Y(t_1))$  is still well defined, meaning that the point  $(t_1, Y(t_1))$  belongs to the rectangle  $\mathcal{R}$ .

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<sup>2</sup>The proof becomes less technical if one makes the more restrictive assumption that  $|F(t, y)| \leq M$  for all  $(t, y_0)$  with  $|t - t_0| \leq a$  and  $y \in \mathbb{R}$ . We then have  $a_* = a$  and much of the discussion in step 1 is then superfluous. We will first discuss this special case in class.



For this we have to check  $|Y(t_1) - y_0| \leq b$ . Indeed we have that  $|Y(t_1) - y_0| = |F(t_0, Y(t_0))(t_1 - t_0)| \leq M(t_1 - t_0) \leq Ma_* \leq b$  by definition of  $a_*$ . A similar calculation has to be made at every step.

To be rigorous we formulate the following

**Claim.** *For  $k = 0, \dots, N$  there are numbers  $y_k$  so that*

$$\begin{aligned} |y_k - y_0| &\leq M(t_k - t_0) \leq b \text{ and} \\ y_k &= y_{k-1} + F(t_{k-1}, y_{k-1})(t_k - t_{k-1}). \end{aligned}$$

For  $k = 0$  the statement is clear. We argue by induction. Above we have just verified this claim for  $k = 1$ , and in a similar way we do the induction step.

If  $k \in \{1, \dots, N-1\}$  we prove the claim for  $k+1$ , i.e. the existence of  $y_{k+1}$  with the required properties, under the induction hypothesis, that  $y_1, \dots, y_k$  have been found. Since by the induction hypothesis  $|y_k - y_0| \leq M(t_k - t_0)$  which is  $\leq b$  the expression  $F(t_k, y_k)$  is well defined and thus  $y_{k+1} = y_k + F(t_k, y_k)(t_{k+1} - t_k)$  is well defined. To check that  $|y_{k+1} - y_0| \leq M(t_{k+1} - t_0)$  we observe that

$$\begin{aligned} |y_{k+1} - y_0| &= |y_{k+1} - y_k + y_k - y_0| \leq |y_{k+1} - y_k| + |y_k - y_0| \\ &= |F(t_k, y_k)(t_{k+1} - t_k)| + |y_k - y_0| \leq M(t_{k+1} - t_k) + |y_k - y_0| \\ &\leq M(t_{k+1} - t_k) + M(t_k - t_0) = M(t_{k+1} - t_0) \end{aligned}$$

where we have in the second to last step used the induction hypothesis. Of course  $M(t_{k+1} - t_0) \leq Ma_* \leq b$ . The claim follows by induction.

Now that the claim is verified we can define  $Y(t)$  on  $[t_0, a_*]$  by  $Y(t_k) = y_k$  and

$$Y(t) = y_k + F(t_k, y_k)(t - t_k), \quad t_k < t < t_{k+1}, \quad k = 0, 1, \dots, N-1.$$

Observe that this definition is also valid for  $t_k \leq t \leq t_{k+1}$ . The function  $Y$  is continuous, piecewise linear, and the absolute values of all slopes are bounded by  $M$ .

**2.** *The function  $Y \equiv Y_\varepsilon$  constructed in part 1 satisfies the inequality*

$$|Y(t) - Y(t')| \leq M|t - t'| \text{ whenever } t, t' \text{ are both in } [t_0, t_0 + a_*].$$

*Proof:* W.l.o.g  $t' < t$ . If  $t', t$  lie in the same partition interval  $[t_k, t_{k+1}]$  then this is immediate since

$$|Y(t) - Y(t')| = |F(t_k, y_k)(t - t')| \leq M|t - t'|.$$

If  $t', t$  lie in different partition intervals,  $t' \in [t_k, t_{k+1}]$ ,  $t \in [t_l, t_{l+1}]$  with  $k < l$ , then

$$\begin{aligned}
|Y(t) - Y(t')| &= \left| Y(t) - Y(t_l) + \sum_{k < \nu < l} Y(t_{\nu+1}) - Y(t_\nu) + Y(t_{k+1}) - Y(t') \right| \\
&\leq |Y(t) - Y(t_l)| + \sum_{k < \nu < l} |Y(t_{\nu+1}) - Y(t_\nu)| + |Y(t_{k+1}) - Y(t')| \\
&= |F(t_l, y_l)|(t - t_l) + \sum_{k < \nu < l} |F(t_\nu, y_\nu)|(t_{\nu+1} - t_\nu) + |F(t_k, y_k)|(t_{k+1} - t') \\
&\leq M(t - t_l) + \sum_{k < \nu < l} M(t_{\nu+1} - t_\nu) + M(t_{k+1} - t') = M(t - t');
\end{aligned}$$

here the middle terms with the sum  $\sum_{k < \nu < l}$  are only present when  $k + 1 < l$ . The claim 2 is proved.

**3.** Note that if we define  $g(t) = F(t_{k-1}, Y(t_{k-1}))$  if  $t_{k-1} \leq t < t_k$  then  $g$  is a step function and  $Y$  is differentiable in the open intervals  $(t_{k-1}, t_k)$  with derivative  $Y'(t) = g(t)$ .

**Claim:** For  $t_0 \leq t \leq t_0 + \tilde{a}$  we have

$$Y(t) = y_0 + \int_{t_0}^t g(s) ds$$

and

$$|g(s) - F(s, Y(s))| \leq \varepsilon \text{ if } t_{k-1} < s < t_k.$$

We first verify the first formula for  $t = t_k$ . Then

$$\begin{aligned}
Y(t_k) - y_0 &= Y(t_k) - Y(t_0) = \sum_{\nu=1}^k [Y(t_\nu) - Y(t_{\nu-1})] \\
&= \sum_{\nu=1}^k F(t_{\nu-1}, Y(t_{\nu-1}))(t_\nu - t_{\nu-1}) = \sum_{\nu=1}^k \int_{t_{\nu-1}}^{t_\nu} g(s) ds = \int_{t_0}^{t_k} g(s) ds
\end{aligned}$$

Similarly for  $t_k < t < t_{k+1}$ ,

$$Y(t) - Y(t_k) = F(t_k, Y(t_k))(t - t_k) = \int_{t_k}^t g(s) ds.$$

We put the two formulas together and get

$$Y(t) = Y(t_k) + \int_{t_k}^t g(s) ds = y_0 + \int_{t_0}^{t_k} g(s) ds + \int_{t_k}^t g(s) ds = y_0 + \int_{t_0}^t g(s) ds.$$

For the second assertion let  $t_{k-1} < s < t_k$  and observe

$$|g(s) - F(s, Y(s))| = |F(t_{k-1}, Y(t_{k-1})) - F(s, Y(s))|$$

and  $|Y(t_{k-1}) - Y(s)| \leq M|t_{k-1} - s| \leq M(t_k - t_{k-1}) \leq \delta/2$  (since the maximal width of the partition is  $< \delta/2M$ ). Thus the distance of the points  $(t_{k-1}, Y(t_{k-1}))$  and  $(s, Y(s))$  is no more than  $\delta$ . It follows that

$$|F(t_{k-1}, Y(t_{k-1})) - F(s, Y(s))| < \varepsilon.$$

**4. Claim:** For  $t_0 \leq s \leq t_0 + a_*$ ,

$$\left| Y(t) - \left( y_0 + \int_{t_0}^t F(s, Y(s)) ds \right) \right| \leq \varepsilon a_*.$$

This follows from part 3, since the right hand side is

$$\begin{aligned} & \left| y_0 + \int_{t_0}^t g(s) ds - \left( y_0 + \int_{t_0}^t F(s, Y(s)) ds \right) \right| \\ &= \left| \int_{t_0}^t [g(s) - F(s, Y(s))] ds \right| \end{aligned}$$

and this is estimated by

$$\int_{[t_0, t]} |g(s) - F(s, Y(s))| ds \leq \varepsilon(t - t_0) \leq \varepsilon a_*.$$

**5.** So far we have only considered a fixed function  $Y \equiv Y_\varepsilon$ . Now consider a sequence of such functions  $Y_{\varepsilon(n)}$  where  $\varepsilon(n) \rightarrow 0$  (for example  $\varepsilon(n) = 2^{-n}$ ). These functions satisfy the properties in part 1-4 with the parameter  $\varepsilon = \varepsilon(n)$  and with  $\delta = \delta(\varepsilon(n))$ .

By part 2 the family of functions  $Y_{\varepsilon(n)}$  is uniformly bounded and uniformly equicontinuous. Indeed for all  $n$  and all  $t \in [t_0, t_0 + a_*]$  we have shown

$$|Y_{\varepsilon(n)}(t) - Y_{\varepsilon(n)}(t')| \leq M|t - t'|$$

(which implies the uniform equicontinuity) and it follows also that

$$|Y_{\varepsilon(n)}(t)| \leq |Y_{\varepsilon(n)}(t_0)| + |Y_{\varepsilon(n)}(t) - Y_{\varepsilon(n)}(t_0)| \leq |y_0| + M|t - t_0| \leq |y_0| + Ma_*.$$

Thus the Arzelà-Ascoli theorem allows us to choose an increasing sequence of integers  $m_n$  such that the subsequence  $Y_{\varepsilon(m_n)}(t)$  converges uniformly on  $[t_0, t_0 + a_*]$  to a limit  $y(t)$ . As a uniform limit of continuous functions  $y$  is continuous on  $[t_0, t_0 + a_*]$ . Now by the uniform continuity of  $F$  and the uniform convergence of  $Y_{\varepsilon(m_n)}$  we see<sup>3</sup> that  $F(t, Y_{\varepsilon(m_n)}(t))$  converges to  $F(t, y(t))$ , uniformly on  $[t_0, t_0 + a_*]$ . This implies that  $y_0 + \int_{t_0}^t F(s, Y_{\varepsilon(m_n)}(s)) ds$  converges to  $y_0 + \int_{t_0}^t F(s, y(s)) ds$ . But by part (4) we also have

$$\left| Y_{\varepsilon(m_n)}(t) - \left( y_0 + \int_{t_0}^t F(s, Y_{\varepsilon(m_n)}(s)) ds \right) \right| \leq \varepsilon(m_n) a_* \rightarrow 0 \text{ as } n \rightarrow \infty.$$

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<sup>3</sup>Provide the details of this argument.

Thus we obtain for the limit (as  $n \rightarrow \infty$ )

$$y(t) = y_0 + \int_{t_0}^t F(s, y(s)) ds$$

which is the desired integral equation for  $y$ . □

*Final remark.* There are many generalizations of Picard's and Peano's theorems. For example the proofs apply, with only very minor changes, to systems of equations with an unknown vector valued function  $(y_1(t), \dots, y_m(t))$ ,

$$y_1'(t) = F_1(t, y_1(t), \dots, y_m(t))$$

$$\vdots$$

$$y_m'(t) = F_m(t, y_1(t), \dots, y_m(t))$$

with initial value conditions

$$y_i(t_0) = y_{i,0}, \quad i = 1, \dots, m.$$