ALL PROBLEMS AND SOLUTIONS

Exercise 1. Let $K : [a, b] \times [a, b] \to \mathbb{R}$ be a differentiable function such that

$$\max_{[a,b]^2} |K(x, t)| \leq 1, \max_{[a,b]^2} \left| \frac{\partial K}{\partial x}(x, t) \right| \leq 1.$$

Consider the space $C[a, b]$ of continuous functions on $[a, b]$ with the sup-norm. For $f \in C[a, b]$, define

$$Af(x) = \int_a^b K(x, t)f(t)\ dt.$$

(1) Prove that $\{Af : \max_{[a,b]} |f(x)| \leq 1\}$ is a totally bounded subset of $C[a, b]$.
(2) If in (1) we drop the assumption $\max_{[a,b]^2} \left|\frac{\partial K}{\partial x}(x, t)\right| \leq 1$ and keep the other assumptions, does $\{Af : \max_{[a,b]} |f(x)| \leq 1\}$ have to be a totally bounded subset of $C[a, b]$?

Solution 1.

(1) The usual definition of a totally bounded subset E of a metric space M is one where for any $\varepsilon > 0$, we can cover the set with finitely many ε-balls centered in E. This turns out to be equivalent to a set being precompact (that is, every sequence in E has a convergent subsequence), which will be a more convenient way to prove a set is totally bounded. We will prove the useful direction: suppose $E \subset M$ is not totally bounded. Then we can find an infinite sequence of ε-seperated points in M, which cannot be convergent, and hence M is not precompact. The contrapositive of what we have proved is that precompact sets must be totally bounded.

So it suffices to prove that $\{Af : \max_{[a,b]} |f(x)| \leq 1\}$ is precompact. Suppose $g_n = Af_n$ for some sequence f_n satisfying $\max_{x\in[a,b]} |f_n(x)| \leq 1$. We need to check two conditions to apply Arzela-Ascoli. First, we needed to check that $\{g_n\}$ is uniformly bounded. This is because $|Af_n(x)| \leq (b-a) \sup_{t\in[a,b]} |K(x, t)f(t)| \leq (b-a)$. Now, we need to prove that $\{g_n\}$ is uniformly equicontinuous. Fix $\varepsilon > 0$. For any $x, y \in [a, b]$ and any n,

$$|Af_n(x) - Af_n(y)| \leq \int_a^b |K(x, t) - K(y, t)|f(t)\ dt.$$

By the mean value theorem, $|K(x, t) - K(y, t)| \leq |x - y| \sup_{(\overline{x},t)\in[a,b]} \left|\frac{\partial K}{\partial x}(\overline{x}, t)\right| \leq |x - y|$, so

$$|Af_n(x) - Af_n(y)| \leq (b - a)|x - y| \sup_{t\in[a,b]} |f(t)| \leq (b - a)|x - y|.$$

Hence g_n is uniformly equicontinuous. By Arzela-Ascoli, it has a convergent subsequence, so $\{Af : \max_{[a,b]} |f(x)| \leq 1\}$ is precompact, as desired.

(2) We will give essentially the same proof, but it will require a little more care without using the bound $\max_{[a,b]^2} \left|\frac{\partial K}{\partial x}(x, t)\right| \leq 1$. The only part that changes is proving equicontinuity. Since K is differentiable, it is continuous, and since $[a, b]$ is compact,

it is uniformly continuous. Fix $\varepsilon > 0$ and choose $\delta > 0$ such that if $||(x,t)-(y,s)|| < \delta$, then $|K(x,t) - K(y,s)| < \varepsilon/(b-a)$. Then

$$|Af_n(x) - Af_n(y)| < \int_a^b |K(x,t) - K(y,t)||f(t)|\ dt \leq \varepsilon/(b-a) \int_a^b |f(t)|\ dt < \varepsilon.$$

We therefore have equicontinuity, and the rest of the proof follows.

Exercise 2. For a sequence (a_k) let $s_n = \sum_{k=1}^n a_k$ and $\sigma_L = \frac{1}{L}\sum_{n=1}^L s_n$. We say that $\sum_{k=1}^\infty a_k$ is Cesáro summable to S if $\lim_{L\to\infty} \sigma_L = S$.

(1) Prove: $s_n - \sigma_n = \frac{(n-1)a_n + (n-2)a_{n-1} + \cdots + a_2}{n}$.
(2) Prove: If $\sum_{k=1}^\infty a_k$ is Cesáro summable to S and if $\lim_{k\to\infty} ka_k = 0$, then $\sum_{k=1}^\infty a_k$ converges and $\sum_{k=1}^\infty a_k = S$.

Solution 2.

(1) Let's induct on n. If $n = 1$, $s_n = \sigma_n = a_1$, so the desired equality holds. Now suppose the equality holds for some n. Note that $\sigma_{n+1} = \frac{n}{n+1}\sigma_n + \frac{1}{n+1}s_{n+1}$. Then $s_{n+1} - \sigma_{n+1} = \frac{n}{n+1}(s_{n+1} - \sigma_n)$. Applying the inductive hypothesis,

$$s_{n+1} - \sigma_n = s_{n+1} - s_n + \frac{(n-1)a_n + \cdots + a_2}{n} = \frac{na_{n+1} + (n-1)a_n + \cdots + a_2}{n}.$$

Then $\frac{n}{n+1}(s_{n+1} - \sigma_n) = \frac{na_{n+1} + (n-1)a_n + (n-2)a_{n-1} + \cdots + a_2}{n+1}$, as desired.

(2) We need to prove that $\lim_{k\to\infty} s_k = S$, so it suffices to prove

$$\lim_{k\to\infty} s_k - \sigma_k = \lim_{k\to\infty} \frac{(k-1)a_k + (k-2)a_{k-1} + \cdots + a_2}{k} = 0.$$

Fix $\varepsilon > 0$ and choose N large enough that $|ka_k| < \varepsilon/2$, and hence $|(k-j)a_k| < \varepsilon/2$ for all $k \geq N$ and $j < k$. Then

$$\frac{(k-1)a_k + \cdots + a_2}{k} = \frac{(k-1)a_k + \cdots + (N-1)a_N}{k} + \frac{(N-2)a_{N-1} + \cdots + a_2}{k}$$

The first expression on the right is $< \varepsilon/2$. Taking $k > N$ sufficiently large makes the second expression $< \varepsilon/2$ as well, since the numerator is fixed. Then the whole sum is $< \varepsilon$, so $|s_k - \sigma_k| < \varepsilon$. Since ε was arbitrary, $\lim_{k\to\infty} s_k - \sigma_k = 0$, and we are done.

Exercise 3. Consider the space $C([0,10])$ of continuous functions on $[0,10]$, and for a given large number L consider the metric $d_L(f,g) = \max_{x\in[0,10]} e^{-Lx}|f(x) - g(x)|$.

(1) Argue that $C([0,10])$ with the metric d_L is a complete metric space.
(2) Show that there is a unique function which is continuous on $[0,10]$ and satisfies

$$f(x) = -15 + \cos(x) \int_0^x e^{e^{tx}} f(t)\ dt$$

for all $x \in [0,10]$.

Solution 3.

(1) Note that d_0 is the usual sup metric on $C([0,10])$, which is well-known to be complete. Also, $d_L(f,g) = d_0(e^{-Lx}f, e^{-Lx}g)$, so d_L must be a well-defined metric. Since $d_0(f,g)e^{-10L} \le d_L(f,g) \le d_0(f,g)$, d_L and d_0 have the same Cauchy sequences and convergent sequence. Since d_0 is complete, d_L must be complete as well.

(2) We need to choose an appropriate value of L so that we can apply the contraction mapping theorem. Let $Tf(x) = -15 + \cos(x)\int_0^x e^{e^{tx}} f(t)\,dt$. Then

$$d_L(Tf,Tg) \le \sup_{x\in[0,10]} \int_0^x e^{e^{tx}-L(x-t)} e^{-Lt}|f(t)-g(t)|\,dt.$$

Suppose we can prove that for any x and functions $f,g \in C([0,10])$,

$$\int_0^x e^{e^{tx}-L(x-t)} e^{-Lt}|f(t)-g(t)|\,dt < \sup_{t\in[0,10]} e^{-Lt}|f(t)-g(t)|,$$

or equivalently, that $\int_0^x e^{e^{tx}-L(x-t)}\,dt < 1$ for any $x \in [0,10]$. If we had this, then since $[0,10]$ is compact, $\sup_{x\in[0,10]} \int_0^x e^{e^{tx}-L(x-t)}\,dt := q < 1$, and hence $d_L(Tf,Tg) \le qd_L(f,g)$, and the contraction mapping theorem gives the desired result. We will then aim to prove that for L sufficiently large, $\int_0^x e^{e^{tx}-L(x-t)}\,dt < 1$.

Integrands can be small because the integrand is small or because the domain of integration is small. We will need to take advantage of both reasons to prove the desired bound, because if $x-t$ can be arbitrarily close to 0, then we have no hope of making $e^{tx} - L(x-t) < 0$, as e^{tx} will approach 1 while $L(x-t)$ will approach 0. Let ε_0 be a small constant, to be determined later. For $x > \varepsilon_0$, we can write $\int_0^x e^{e^{tx}-L(x-t)}\,dt = \int_0^{x-\varepsilon_0} e^{e^{tx}-L(x-t)}\,dt + \int_{x-\varepsilon_0}^x e^{e^{tx}-L(x-t)}\,dt$. The latter integral is bounded above by $\varepsilon_0 e^{e^{100}}$, so taking ε_0 sufficiently small, we can ensure that it is less than $1/2$. This will be our choice for ε_0. Now, we will choose L large enough to bound the first integral. We know that $x-t > \varepsilon_0$, so $L(x-t) > L\varepsilon_0$. On the other hand $e^{tx} \le e^{100}$. If we take L large enough, $e^{100} - L\varepsilon_0 < -\log(20)$, so $e^{tx} - L(x-t) < -\log(20)$, and hence $e^{e^{tx}-L(x-t)} < 1/20$. Then $\int_0^{x-\varepsilon_0} e^{e^{tx}-L(x-t)}\,dt < (x-\varepsilon_0)/20 < 1/2$. Thus, $\int_0^x e^{e^{tx}-L(x-t)}\,dt < 1$ if $x > \varepsilon_0$. If $x \le \varepsilon_0$, then $\int_0^x e^{e^{tx}-L(x-t)}\,dt \le \varepsilon_0 e^{e^{100}} < 1/2$. Either way, $\int_0^x e^{e^{tx}-L(x-t)}\,dt < 1$, so we are done.

Exercise 4. Let

$$\Omega = \{(x_1,x_2) \in \mathbb{R}^2 : 0 < x_2 < x_1^2 \le 1/2\}.$$

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x) = (x_1^2 + x_2^2)^{-b/2}|\log(x_1^2+x_2^2)|^{-\gamma}.$$

Determine for which values $b > 0, \gamma \in \mathbb{R}$, $\int_\Omega f(x)\,dx$ is finite.

Solution 4. Let's radially integrate. The only part of f that could cause the integral to diverge is the singularity at 0 (depending on the sign of γ, there might be a singularity where $x_1^2 + x_2^2 = 1$ as well, but we can never get very close to it because $\Omega \subset B_{3/4}(0)$), so we are free to ignore parts of the domain outside of the circle of radius $1/2$. We also have that both Ω and f are symmetric about the x_2-axis, so $\int_\Omega f(x)\,dx < \infty$ if and only if $\int_{\Omega^+} f(x)\,dx > 0$, where $\Omega^+ = \{(x_1, x_2) \in \Omega : x_1 > 0\}$. We have

$$\int_{\Omega^+} f(x)\, dx = \int_0^{1/2} \frac{r^{-b}}{|\log(r)|^\gamma} M(r)\, dr,$$

where $M(r)$ denotes the Lebesgue measure of the set

$$A(r) := \{\theta \in [0, \pi/2] : (r\cos(\theta), r\sin(\theta)) \in \Omega\} = \{\theta \in [0, \pi/2) : r^2\cos^2(\theta) > r\sin(\theta) > 0\}$$

Let's prove that $M(r) \approx r$ ($a \approx b$ means $ca < b < Ca$ for positive constants c, C), justifying replacing $M(r)$ in our integral with r. Applying the Pythagorean identity, we see that $A(r) = [0, \xi)$, where $u = \sin(\xi)$ solves $ru^2 + u - r = 0$, and hence $M(r) = \xi$. Using the quadratic formula, we have $\sin(\xi) = \frac{\sqrt{1+4r^2}-1}{2r}$. Since $\sin(\xi) \leq \xi \leq 2\sin(\xi)$ for $\xi \in [0, \pi/2]$, we know that $M(r) \approx \frac{\sqrt{1+4r^2}-1}{2r}$. Taylor expanding $r \mapsto \sqrt{1+4r^2}$, we have $\sqrt{1+4r^2} = 1 + 2r^2 + O(r^4)$. Since $r \in [0, 1/2]$, $2r^2 + O(r^4) \approx r^2$, and hence $M(r) \approx \frac{r^2}{r} = r$, as desired.

Substituting this in, we see that the original integral converges if and only if

$$\int_0^{1/2} \frac{r^{1-b}}{|\log(r)|^\gamma}\, dr < \infty.$$

For me, it is more comfortable to first substitute $u = 1/r$. The integral becomes $\int_2^\infty \frac{u^{b-3}}{|\log(u)|^\gamma}\, dr$ and now the singularity is at ∞. Any (positive) power of u grows faster than any power of $\log(u)$, so if $b - 3 < -1$, the integral converges and if $b - 3 > -1$, it diverges. Equivalently, if $b < 2$, the integral converges and if $b > 2$ it diverges, no matter what γ is. On the other hand, if $b = 2$, then the integral becomes $\int_2^\infty \frac{1}{u|\log(u)|^\gamma}\, du$. Substituting $v = \log(u)$, this becomes $\int_{\log(2)}^\infty \frac{1}{v^\gamma}\, dv$, which is finite if and only if $\gamma > 1$.

Exercise 5. Suppose $f_n \in L^1(\mathbb{R})$ and the sequence ε_n of positive real numers satisfies $\lim_{n\to\infty} \varepsilon_n = 0$. Define the sets $E_n = \{x : |f_n(x)| \geq \varepsilon_n\}$ and assume that $\sum_n m(E_n) < \infty$. Prove that

(1) $\lim_{n\to\infty} f_n = 0$ Lebesgue a.e. on $\mathbb{R}$.
(2) For every $\delta > 0$, there is a set Ω such that $m(\Omega) < \delta$ and $\lim_{n\to\infty} f_n = 0$ uniformly on $\mathbb{R} \setminus \Omega$.

Solution 5.

(1) By Borel-Cantelli, $\sum_n m(E_n) < \infty$ implies that for almost every x, $x \in E_n$ finitely often (in other words, $m(\limsup E_n) = 0$). Then for any x and for all n sufficiently large, $x \notin E_n$, and hence $|f_n(x)| < \varepsilon_n$. Since $\varepsilon_n \to 0$, $f_n(x) \to 0$ as well.
(2) Since $\sum_n m(E_n) < \infty$, we know that $\lim_{N\to\infty} \sum_{n=N}^\infty m(E_n) = 0$. Then for any δ, we can find N sufficiently large such that $\sum_{n=N}^\infty m(E_n) < \delta$. Set $\Omega = \bigcup_{n\geq N} E_n$. Then by subadditivity, $m(\Omega) < \delta$. For any $x \in \mathbb{R} \setminus \Omega$, we have $f_n(x) < \varepsilon_n$ for any $n \geq N$. Therefore, f_n converges uniformly to 0 on $\mathbb{R} \setminus \Omega$.

Exercise 6. Let $E \subset \mathbb{R}$. Suppose $g, f_n \in L^1(E)$, $\sup_n ||f_n||_{L^1} < \infty$ and $\lim_{n\to\infty} f_n = 0$ in Lebesgue measure. Prove that

$$\lim_{n\to\infty} \int_E \sqrt{|f_n g|}\, dx = 0.$$

Solution 6. We will use three different sources of "smallness" to control $\int_E \sqrt{|f_n g|(x)}\ dx$. One will be the convergence of f_n in measure. In other words, for any $\varepsilon > 0$ and any $\delta > 0$, for n sufficiently large, $|f_n|(x) > \varepsilon$ only occurs on a set of measure $< \delta$. The other two will come from the fixed function g.

I will prove both of the facts, although I would be surprised if you lost points for not doing so on the qual itself. Since $g \in L^1$, we know that for any $\varepsilon > 0$, we can find a finite measure set $F \subset E$ such that $\int_{E-F} |g|(x)\ dx < \varepsilon$. This is fairly easy to prove: if we define $F_x = E \cap [-x, x]$, then by the monotone convergence theorem, $\lim_{x\to\infty} \int_{F_x} |g|(y)\ dy = \int_E |g|(y)\ dy$, so we can find some x sufficiently large such that $\int_{E-F_x} |g|(y)\ dy < \varepsilon$. Our final fact is that for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $m(F) < \delta$, then $\int_F |g|(x)\ dx < \varepsilon$. Suppose this is not the case. Then there exists $\varepsilon > 0$ such that we can find sets $F_1, F_2, \ldots$ with $m(F_k) < 2^{-k}$ and $\int_E \chi_{F_k} |g|(x)\ dx > \varepsilon$. We know that $\lim_{k\to\infty} \chi_{F_k} = 0$ outside of a set of measure 0, so by dominated convergence, $\lim_{k\to\infty} \int_E \chi_{F_k} |g|(x)\ dx = 0$, contradicting $\int_E \chi_{F_k} |g|(x)\ dx > \varepsilon$ for all k.

With these three sources of smallness in hand, we will complete the proof. Let $M = \sup_m ||f_n||_{L^1}$. Fix $\varepsilon > 0$ and choose a set F such that $\int_F |g(x)|\ dx < \varepsilon^2/(9M)$ and $m(F^c) < \infty$. Choose $\delta > 0$ such that for any $G \subset E$ with $m(G) < \delta$, $\int_G |g(x)|\ dx < \varepsilon^2/(9M)$. Finally, for n sufficiently large, choose sets $H_n \subset F^c$ with $m(H_n) < \delta$ such that for all $x \in E - H_n$, $|f_n(x)| < \varepsilon^2/(9m(F^c)||g||_{L^1})$. Write

$$\int_E \sqrt{|f_n g|(x)}\ dx = \int_F \sqrt{|f_n g|(x)}\ dx + \int_{H_n} \sqrt{|f_n g|(x)}\ dx + \int_{F^c - H_n} \sqrt{|f_n g|(x)}\ dx.$$

We will bound all three integrals using Cauchy-Schwarz. The first is bounded above by

$$\left(\int_F |f_n|(x)\ dx \int_F |g|(x)\ dx\right)^{1/2} < (||f_n||_{L^1}\varepsilon/(9M))^{1/2} < \varepsilon/3.$$

The second is bounded above by

$$\left(\int_{H_n} |f_n|(x)\ dx \int_{H_n} |g|(x)\ dx\right)^{1/2} < (||f_n||_{L^1}\varepsilon^2/(9M))^{1/2} < \varepsilon/3.$$

The third is bounded above by

$$\left(\int_{F^c - H_n} |f_n|(x)\ dx \int_{F^c - H_n} |g|(x)\ dx\right)^{1/2} < (m(F^c)\varepsilon^2/(9m(F^c)||g||_{L^1})||g||_{L^1})^{1/2} < \varepsilon/3.$$

Hence, $\int_E \sqrt{|f_n g|(x)}\ dx < \varepsilon$ for n sufficiently large. Since ε was arbitrary, we are done.

Exercise 7. Suppose that on a set E of finite measure, $f_n \to f$ in measure and $g_n \to g$ in measure and f is finite a.e.. Prove that $f_n g_n \to fg$ in measure on E.

Hint: Can you prove $f_n^2 \to f^2$ in measure?

Solution 7. Suppose we can prove the hint. Then $(f_n + g_n)^2 \to (f+g)^2$, $f_n^2 \to f^2$, and $g_n^2 \to g^2$ in measure. It follows that $f_n g_n = \frac{(f_n+g_n)^2 - f_n^2 - g_n^2}{2} \to \frac{(f+g)^2 - f^2 - g^2}{2} = fg$ in measure, as desired.

It remains to prove the hint. Fix $\varepsilon > 0$. We know that $m(\{x : |f_n - f|(x) > \varepsilon^{1/2}\}) \to 0$, so $m(\{x : |f_n - f|(x)^2 > \varepsilon\}) \to 0$. Since ε was arbitrary, we therefore know that $|f_n - f|^2 \to 0$ in measure. We can expand this to $f_n^2 + f^2 - 2f_n f \to 0$. Then if we can prove $f_n f \to f^2$, we would have that $f_n^2 = f_n^2 + f^2 - 2f_nf + 2f_nf - f^2 \to f^2$ in measure, since $f_n^2 + f^2 - 2f_nf \to 0$ in measure and $2f_nf - f^2 \to f^2$ in measure.

So let's prove that $f_nf \to f^2$ in measure. Fix $\varepsilon > 0$ and $\delta > 0$, we need to find $N \in \mathbb{N}$ such that if $n \geq N$, then $m(\{x : |f_nf - f^2| > \varepsilon\}) < \delta$. Since f is finite a.e., for some K sufficiently large, $|f|(x) \leq K$ for all x in a set A such that $m(E \setminus A) < \frac{\delta}{2}$. Now, choose N sufficiently large so that $m(\{x : |f_n - f| > \frac{\varepsilon}{K}\}) < \frac{\delta}{2}$ for all $n \geq N$. This is possible because $f_n \to f$ in measure. Then for $n \geq N$,

$$\begin{aligned} m(\{x : |f_nf - f^2| > \varepsilon\}) &< m(E \setminus A) + m(\{x \in A : |f_nf - f^2| > \varepsilon\}) \\ &\leq \frac{\delta}{2} + m(\{x \in A : |f_n - f| > \frac{\varepsilon}{|f|}\}) \\ &\leq \frac{\delta}{2} + m(\{x \in A : |f_n - f| > \frac{\varepsilon}{K}\}) \\ &< \delta. \end{aligned}$$

Since δ, ε was arbitrary, we know that $m(\{x : |f_nf - f^2|(x) > \varepsilon\}) \to 0$ for all $\varepsilon > 0$, so $f_nf \to f^2$ in measure, completing the proof.

Exercise 8. Let $X = \{P : \mathbb{R} \to \mathbb{R} | P \text{ is a polynomial}\}$. Prove that there does not exist a norm $||\cdot||$ on X such that $(X, ||\cdot||)$ is a Banach space.

Solution 8. Suppose such a norm $||\cdot||$ exists. Let $||P||_{\text{coeff}}$ be the sup-norm of the coefficients of P. Let $K_{n,m} = \overline{\{P \in X : \deg(P) \leq n, ||P||_{\text{coeff}} \leq m\}}$. Since $\bigcup_{n,m} K_{n,m} = X$, by the Baire category theorem, some $K_{n,m}$ must have non-empty interior. Then in particular, there exists some open ball $B(P, r) \subset K_{n,m}$. This contains an element of degree $\geq n+1$ given by $Q = P + \sigma x^{n+1}$ for σ sufficiently small. Then we can write $Q = \lim_{k\to\infty} P_k$ for elements $P_k \in K_{n,m}$. Write $P_k = \sum_{j=0}^n a_{j,k}x^j$. Since each $a_{j,k}$ falls in a compact set, we may pass to a subsequence so that $a_{j,k} \to a_j$ for each j. Let $R(x) = \sum_{j=0}^n a_jx^j$. Then $P_k \to R$. But $R \neq Q$, since $\deg(R) \leq n$ and $\deg(Q) \geq n+1$, a contradiction.

Exercise 9. Suppose $g_n \in S(\mathbb{R}^2)$ and $\lim_{n\to\infty} ||g_n||_{L^2}(\mathbb{R}^2) = 0$. Show that there are $f_n \in C^2(\mathbb{R}^2)$ such that $\Delta f_n = f_n + g_n$ and f_n satisfies

(1) $\lim_{n\to\infty} f_n(0,0) = 0$.
(2) $\lim_{n\to\infty} ||\partial^2_{x_1x_2}(f_n)||_{L^2(\mathbb{R}^2)} = 0$.

Solution 9. Before we prove the properties, let's solve the equation $\Delta f_n = f_n + g_n$. Take the Fourier transform of both sides. Hopefully you recall that $\widehat{\Delta f_n}(\xi) = -|\xi|^2\hat{f}_n(\xi)$, so the desired equation becomes $|\xi|^2\hat{f}_n(\xi) + \hat{f}_n(\xi) = -\hat{g}(\xi)$, or in other words, $\hat{f}_n = -\frac{\hat{g_n}(\xi)}{1+|\xi|^2}$. Since g_n is Schwartz, $\hat{f}_n$ is Schwartz as well, and hence it's Fourier inverse f_n is a Schwartz function and hence in C^2. By construction, these functions f_n satisfy $\Delta f_n = f_n + g_n$.

(1) Let $C = ||\frac{1}{1+|\xi|^2}||_{L^2}(\mathbb{R}^2)$. Then

$$f_n(0) = -\int e^{i0\cdot\xi}\frac{\hat{g_n}(\xi)}{1+|\xi|^2}\,dx = -\int \frac{\hat{g_n}(\xi)}{1+|\xi|^2}\,dx \leq ||\hat{g_n}||_{L^2}||\frac{1}{1+|\xi|^2}||_{L^2} = C||\hat{g_n}||_{L^2} = C||g_n||_{L^2}.$$

Since $||g_n||_{L^2} \to 0$, $f_n(0) \to 0$ as well.

(2) By Plancheral's theorem,

$$||\partial^2_{x_1x_2} f_n||_{L^2(\mathbb{R}^2)} = ||\xi_1\xi_2 \hat{f}_n||_{L^2(\mathbb{R}^2)} = ||\frac{\xi_1\xi_2}{1+|\xi|^2}\hat{g}_n||_{L^2(\mathbb{R}^2)}$$

By Hölder's inequality,

$$||\frac{\xi_1\xi_2}{1+|\xi|^2}\hat{g}_n||_{L^2(\mathbb{R}^2)} \le ||\frac{\xi_1\xi_2}{1+|\xi|^2}||_{L^\infty(\mathbb{R}^2)}||\hat{g}_n||_{L^2(\mathbb{R}^2)} = ||\frac{\xi_1\xi_2}{1+|\xi|^2}||_{L^\infty(\mathbb{R}^2)}||g_n||_{L^2(\mathbb{R}^2)}.$$

Since $(\xi_1-\xi_2)^2 \ge 0$, $\xi_1^2+\xi_2^2 \ge 2\xi_1\xi_2$, and hence $\frac{\xi_1\xi_2}{1+\xi_1^2+\xi_2^2} < 1$. Thus, $||\frac{\xi_1\xi_2}{1+|\xi|^2}||_{L^\infty(\mathbb{R}^2)} < 1$, so $||\partial^2_{x_1x_2} f_n||_{L^2(\mathbb{R}^2)} \le ||g_n||_{L^2}$. Since $||g_n||_{L^2} \to 0$, $||\partial^2_{x_1x_2} f_n||_{L^2(\mathbb{R}^2)} \to 0$ as well.

Exercise 10. The following distributions u, v on $\mathbb{R}^2$ are defined by pairing with Schwartz functions via

$$\langle u, \phi\rangle = \int_0^2 \phi(0,t)\ dt$$

$$\langle v, \phi\rangle = \int_0^2 \phi(t,0)\ dt$$

Show that the convolution $u*v$ can be identified with a finite, absolutely continuous measure μ. Find $g \in L^1(\mathbb{R}^2)$ such that $\int \phi\ d\mu = \int \phi g\ dx$ for all Schwartz functions ϕ.

Solution 10. The primary difficult in this problem for me was remembering what the convolution of two distributions is. In general, if you want to figure out some sort of property for distributions, go to a different page and see if you can figure out the property assuming your distribution is a smooth function. Once you get the property for smooth functions, assume it works for all distributions, then go to the page with the problem and write the property down without proof like you knew it all along.

In this case, if we imagined u and v to be smooth functions, then applying the usual convolution formula and changing variables, we get $\langle u*v, \phi\rangle = \int u(x)v(y)\phi(x+y)\ dx\ dy$. This suggests that we should claim $\langle u*v,\phi\rangle = \langle u(x), \langle v(y), \phi(x+y)\rangle\rangle$, where $\langle v(y), \phi(x+y)\rangle$ is the map $x \mapsto \langle v(y), \phi(x+y)\rangle$. This is well-defined and characterizes the convolution of compactly supported distributions - it might not be how one would define the convolution, but I think you would be justified in immediately writing $\langle u*v,\phi\rangle = \langle u(x), \langle v(y), \phi(x+y)\rangle\rangle$.

With this definition in hand, the rest of the problem is fortunately pretty easy. We can compute $\langle v(y), \phi(x+y)\rangle = \int_0^2 \phi(t+x_1, x_2)\ dt$. This is a smooth function of x, call it $\psi(x)$. Then

$$\langle u(x), \psi(x)\rangle = \int_0^2 \psi(0,s)\ ds = \int_0^2\int_0^2 \phi(t,s)\ dt\ dx = \langle \chi_{[0,2]\times[0,2]}, \phi\rangle.$$

Hence, $\langle u*v, \phi\rangle = \langle g, \phi\rangle$, where $g(x) = \chi_{[0,2]\times[0,2]} \in L^1(\mathbb{R}^2)$, and we can identify $u*v$ with $g(x)\ dx$, which is a finite, absolutely continuous measure.

Exercise 11. Suppose that E is a measurable set of real numbers with arbitrarily small periods (that is, there exists a sequence of real numbers $p_i \to 0$ such that $E + p_i = E$). Prove that either E or it's complement has measure 0.

Solution 11. Suppose $m(E^c) \neq 0$, let's prove $m(E) = 0$. If $m(E) > 0$, then E has a Lebesgue point x. Since $m(E^c) \neq 0$, there is an interval, $[a, a+1]$ such that $\varepsilon = m(E^c \cap [a, a+1]) > 0$. Since x is a Lebesgue point of E, there exists $\delta > 0$ such that if $\delta' < \delta$, then $\frac{m([x-\delta', x+\delta'] \cap E)}{2\delta'} > 1 - \varepsilon/2$. Choose a period p_i such that $p_i < \min(\delta, \varepsilon/12)$. Then we can cover $[a, a+1]$ with intervals $[x_1 - p_i, x_1 + p_i], [x_2 - p_i, x_2 + p_i], \ldots, [x_m - p_i, x_m + p_i]$, where $x_i = x + n_i p_i$ for some $n_i \in \mathbb{Z}$ and $[x_i - p_i, x_i + p_i] \cap [a, a+1] \neq \emptyset$. It follows that $m \geq \frac{1}{2p_i}$. Since Lebesgue measure is translation invariant,

$$m([x_1 - p_i, x_1 + p_i] \cap E) = m(([x - p_i, x + p_i] + np_i) \cap E) = m([x - p_i, x + p_i] \cap E) > 2p_i(1 - \varepsilon/2).$$

We know $2mp_i(1-\varepsilon/2) < m\left(\bigcup_{i=1}^m [x_i - p_i, x_i + p_i] \cap E\right)$ and $\bigcup_{i=1}^m [x_i - p_i, x_i + p_i] \setminus [a, a+1]$ consists of two intervals $[\alpha, a]$ and $[a+1, \beta]$ of length at most p_i, so $m\left(\bigcup_{i=1}^m [x_i - p_i, x_i + p_i] \cap E\right) - m(E \cap [a, a+1]) < 2p_i$, and hence

$$m(E \cap [a, a+1]) > 2mp_i(1 - \varepsilon/2) - 2p_i > (1 - \varepsilon/2) - 2p_i > 1 - 2\varepsilon/3.$$

It follows that $m(E^c \cap [a, a+1]) < \varepsilon/3$, contradicting our definition $m(E^c \cap [a, a+1]) = \varepsilon$.

Exercise 12. Let $\mathbf{T} \subset \mathbb{C}$ be the unit circle. We say that $G \subset \mathbf{T}$ is a subgroup of $\mathbf{T}$ if $1 \in G$, $\zeta_1, \zeta_2 \in G$ implies $\zeta_1\zeta_2 \in G$ and $\zeta_1 \in G$ implies $\zeta_1^{-1} \in G$.

(1) What are the compact subgroups of $\mathbf{T}$?
(2) Give an example of an infinite subgroup $G \subsetneq \mathbf{T}$.
(3) Prove or give a counterexample: there are no measurable subgroups $G \subsetneq \mathbf{T}$ with $|G| > 0$.

Solution 12.

(1) We know $\mathbf{T}$ is a compact subgroup, as are the nth roots of unity $F_n = \{e^{2\pi i k/n} : k = 0, 1, \ldots, n-1\}$. We will show that these are the only compact subgroups.

First, any finite subgroup $G \subset \mathbf{T}$ must be a collection F_n for some n, because it must have some element $e^{2\pi i(1/x)}$ closest to 1. If x is not an integer, let n be the least integer greater than x. Then $0 < \frac{n}{x} - 1 < \frac{1}{x}$, since otherwise would contradict the minimality of n, and therefore $e^{2\pi i n/x}$ is closer to 1 than $e^{2\pi i 1/x}$, a contradiction. Therefore, x must be an integer. If x is an integer n, then $G \supset F_n$, and if there exists some element $e^{2\pi i\theta} \in \mathbf{T} \setminus F_n$, then $\theta - k/n < 1/n$ for some choice of k, in which case $e^{2\pi i(\theta - k/n)}$ is closer to 1 than $e^{2\pi i(1/x)}$, a contradiction. Hence, $G = F_n$.

If G is infinite, then since $\mathbf{T}$ is compact, it has a limit point $\xi \in \mathbf{T}$, and since G is closed, $\xi \in G$. It follows that G contains a sequence of elements ξ_i converging to ξ, in which case $e^{2\pi i\theta_i} = \frac{\xi_i}{\xi} \in G$ is a sequence of elements converging to 1. Let's prove that G is dense in $\mathbf{T}$. For any $e^{2\pi i\eta} \in \mathbf{T}$, we can find m_i such that $|\eta - m_i\theta_i| < \theta_i$. We know that $e^{2\pi i m\theta_i} \in G$ for all $m \in \mathbb{Z}$ and $\lim_{i\to\infty} \theta_i = 0$, so $\lim_{i\to\infty} m_i\theta_i = \eta$ and hence $e^{2\pi i\eta} \in \overline{G}$. But since G is closed, we have that $e^{2\pi i\eta} \in G$, and since $e^{2\pi i\eta}$ was an arbitrary element of $\mathbf{T}$, we have that $G = \mathbf{T}$.

(2) Take $G = \{e^{2\pi i q} : q \in \mathbb{Q}\}$. This is a subgroup, because $\mathbb{Q}$ is a group and it is clearly infinite.

(3) This is true. Let G be a subgroup of $\mathbf{T}$ with $|G| > 0$. First, let's prove that G contains an interval. We could do this using Young's inequality, but I put this on the Lebesgue differentiation theorem worksheet so I guess I will use the Lebesgue differentiation theorem to do this. Let's consider G as subset of $[-\pi, \pi)$ closed under addition. Let θ be a Lebesgue point for G, and without loss of generality, we may assume that

$\theta \in [0, \pi/2]$ (the set of Lebesgue points of G is closed under translation by elements of G, and the argument in part 1 tells us that G contains elements arbitrarily close to 0, so we can repeatedly translate θ by those elements until it is where we want it, it will be convenient to do given our mapping of G onto an interval). Then there exists $\delta > 0$ such that $\frac{m(G\cap[\theta-r,\theta+r])}{2r} > .99$ for all $r \leq \delta$. Suppose there exists $\eta \in [2\theta-\delta/2, 2\theta+\delta/2]\setminus G$. For any $\xi \in [\theta-\delta/2, \theta+\delta/2]$, we know one of ξ and $\eta-\xi$ cannot be in G. Now $\xi, \eta-\xi \in [\theta-\delta, \theta+\delta]$, so for all $\eta - G\cap[\theta-\delta/2, \theta+\delta/2] \subset [\theta-\delta, \theta+\delta]\setminus G$. We know that $m(G\cap[\theta-\delta/2, \theta+\delta/2]) > .99\delta$, so $m(\eta-G\cap[\theta-\delta/2, \theta+\delta/2]) > .99\delta$, and hence $m([\theta-\delta, \theta+\delta]\setminus G) > .99\delta$, so $m([\theta-\delta, \theta+\delta]\cap G) < 1.01\delta$, contradicting our assumption that $m([\theta-\delta, \theta+\delta]\cap G) > 1.98\delta$. Hence, G contains an interval. As previously discussed, G contains arbitrarily small translations, so we can translate the interval in G to cover all of $\mathbf{T}$. Hence, if G has positive measure, then it cannot be strictly contained in $\mathbf{T}$, as desired.

Exercise 13. Let $\{a_n\}$ be a convergent sequence of complex numbers and let $\lim_{n\to\infty} a_n = L$. Let

$$c_n := \frac{1}{n^5}\sum_{k=1}^{n} k^4 a_k.$$

Prove that c_n converges and determine its limit.

Solution 13. By comparing with the integral x^4, we see that $\sum_{k=1}^{n} k^4$ is between $(n+1)^5/5$ and $n^5/5$, so a reasonable guess for the limit would be $L/5$. Let's see if we can prove this directly. We see that

$$|c_n - L/5| = \frac{1}{n^5}\sum_{k=1}^{n}|k^4 a_k - k^4 L| + \frac{1}{n^5}|n^5 L/5 - \sum_{k=1}^{n} k^4 L|. \tag{1}$$

The second sum is more easily bounded: the integral comparison test tells us it must be between 0 and $L\frac{n^5-(n+1)^5}{5n^5}$. Since $n^5 - (n+1)^5 = O(n^4)$ by the binomial expansion, the second integral is $O(1/n)$ and hence goes to 0.

For the first sum, fix $\varepsilon > 0$ and assume $|a_k - L| < \varepsilon/2$ for all $k \geq N$. Then for $n \geq N$, the first sum breaks up into

$$\frac{1}{n^5}\sum_{k=1}^{N}|k^4 a_k - k^4 L| + \frac{1}{n^5}\sum_{k=N+1}^{n} k^4 |a_k - L|. \tag{2}$$

Taking n sufficiently large, the first sum in (2) is $< \varepsilon/3$. Using the integral bound again, the second sum in (2) is $< \varepsilon/2\frac{1}{n^5}\sum_{k=N+1}^{n} k^4 < \varepsilon/2\frac{(n+1)^5}{n^5}$. For n sufficiently large, $\frac{(n+1)^5}{n^5} < 3/2$, so the second sum is less than $2\varepsilon/3$. Hence, both sums on the right side of (1) go to 0 as n goes to ∞, so the limit converges to the desired bound.

Exercise 14. Let $K : [a, b]\times[a, b] \to \mathbb{R}$ be a continuous function. Consider the space $C[a, b]$ of continuous functions on $[a, b]$ with the sup-norm. For $f \in C[a, b]$, define

$$S_K f(x) = \int_a^b K(x, t) f(t)\, dt.$$

(1) Is $\{S_K f : \max_{[a,b]} |f(x)| \leq 1\}$ necessarily a totally bounded subset of $C[a, b]$?

(2) Let f_n be a sequence of continuous functions on $[a,b]$ satisfying

$$\sup_n \sup_{x\in[a,b]} |f_n(x)| \le 1.$$

Does the sequence $S_K f_n$ necessarily have a convergent subsequence? Give a proof or counterexample.

(3) Let K_n be a sequence of continuous functions on $[a,b]\times[a,b]$ and assume that

$$\sup_n \max\{|K_n(x,y)| : (x,y)\in[a,b]\times[a,b]\} \le 1.$$

Let $f\in C[a,b]$. Does the sequence $S_{K_n} f$ necessarily have a convergent subsequence in $C[a,b]$? Prove or give a counterexample.

Solution 14.

(1) Yes. It suffices to prove any sequence in $E = \{S_K f : \max_{[a,b]} |f(x)| \le\}$ has a convergent subsequence, which will follow from the Arzela-Ascoli theorem. Take a sequence $S_K(f_1), S_K(f_2), \cdots \in E$. We need to prove this sequence is uniformly bounded and equicontinuous. Since K is continuous on a compact set, it is bounded above by some M. Then since each $\sup_{x\in[a,b]} |f_n(x)| \le 1$, $\sup_{x\in[a,b]} |S_K(f_n)|(x) \le (b-a)M$. To see equicontinuity, fix $\varepsilon > 0$. Since K is continuous on a compact interval, it is uniformly continuous, so we can find δ such that if $|x-y|<\delta$, then $|K(x,t)-K(y,t)| < \varepsilon/(b-a)$. We know that

$$|S_K(f_n)(x) - S_K(f_n)(y)| \le \int_a^b |K(x,t)-K(y,t)| f(t)\, dt < \varepsilon.$$

By Arzela-Ascoli, $S_K(f_n)$ has a convergent subsequence. It follows that $\{S_K f : \max_{[a,b]} |f(x)| \le 1\}$ must be totally bounded.

(2) Yes, as was proven in part (i).

(3) Without loss of generality, we may assume that $[a,b] = [-1/2, 1/2]$. Set $K_n(x,t) = \cos(nxt)$ and $f(x) = 1$. It is straightforward to compute that

$$S_{K_n} f(x) = \begin{cases} \frac{\sin(nx)}{nx} & x \ne 0 \\ 1 & x = 0 \end{cases}.$$

Suppose $\{S_{K_n} f(x)\}$ had a convergent subsequence. Then its limit f is continuous, but we have $f(0) = 1$ and $f(x) = 0$ for any $x \ne 0$, a contradiction. Hence, $\{S_{K_n} f\}$ cannot have a convergent subsequence.

Exercise 15. Show that for $\alpha \ne 0$,

$$\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{\alpha}{\alpha^2+n^2} = \frac{e^{2\pi\alpha}+1}{e^{2\pi\alpha}-1}.$$

Hint: Apply the Poisson summation formula to the function $f(x) = e^{-c|x|}$, for an appropriate choice of c.

Solution 15. First, let's compute the Fourier transform of $f_c(x) = e^{-2\pi c|x|}$ (using $2\pi c$ instead of c will be convenient for what comes next). We see that

$$\begin{aligned}\hat{f}(\xi) &= \int e^{-2\pi i x\xi} e^{-2\pi c|x|}\ dx \\ &= \int_0^\infty e^{-2\pi x(i\xi+c)}\ dx + \int_{-\infty}^0 e^{-2\pi x(2\pi i\xi - c)}\ dx \\ &= \frac{1}{2\pi}\left(\frac{1}{i\xi+c} - \frac{1}{i\xi-c}\right) \\ &= \frac{1}{\pi}\frac{c}{\xi^2+c^2}\end{aligned}$$

This looks pretty promising. The Poisson summation formula tells us that

$$\sum_{n\in\mathbb{Z}} e^{-2\pi c|n|} = \frac{1}{\pi}\sum_{n\in\mathbb{Z}} \frac{c}{n^2+c^2}.$$

Let's set $c = \alpha$ and prove that $\sum_{n\in\mathbb{Z}} e^{-2\pi\alpha|n|} = \frac{e^{2\pi\alpha}+1}{e^{2\pi\alpha}-1}$. To do so, we will break up the sum and then apply the geometric series formula:

$$\sum_{n\in\mathbb{Z}} e^{-2\pi\alpha|n|} = 2\sum_{n=0}^\infty e^{-2\pi\alpha n} - 1 = \frac{2}{1-e^{-2\pi\alpha}} - 1 = \frac{1+e^{-2\pi\alpha}}{1-e^{-2\pi\alpha}} = \frac{e^{2\pi\alpha}+1}{e^{2\pi\alpha}-1}.$$

Exercise 16. Prove that there are two functions $f_1, f_2 \in C[0,1]$ that solve the following system of equations for all $x \in [0,1]$,

$$20f_1(x) + 3f_2(x) = \sin(x) + \int_0^1 \sin(xt)\sin(f_1(t))\ dt$$

$$-f_1(x) + 10f_2(x) = \cos(x) - \int_0^{1/2} \cos(xt)\cos(f_2(t))\ dt.$$

Solution 16. This is a contraction mapping problem, which means we want to reformulate it into finding a fixed point of an operator $T : M \to M$ for some complete metric space M. Define $A = \begin{bmatrix} 20 & 3 \\ -1 & 10 \end{bmatrix}$ and

$$T_1 f(x) = \sin(x) + \int_0^1 \sin(xt)\sin(f_1(t))\ dt, \text{ and } T_2 f(x) = \cos(x) - \int_0^{1/2} \cos(xt)\cos(f(t))\ dt.$$

We want to solve $A[f_1, f_2]^T = [T_1 f_1, T_2 f_2]^T$. Since A is invertible (it's determinant is 203), this is equivalent to finding a fixed point of the operator $T(f_1, f_2) = A^{-1}[T_1 f_1, T_2 f_2]^T$, where $T : C_0(\mathbb{R})^2 \to C_0(\mathbb{R})^2$ and $C_0(\mathbb{R})^2$ is equppied with the sup-norm for each element: $d((f_1, f_2), (g_1, g_2)) = \sup_{x\in[0,1]} |f_1(x) - g_1(x)| + \sup_{x\in[0,1]} |f_2(x) - g_2(x)|$. To prove that $d(T(f_1, f_2), T(g_1, g_2)) < d((f_1, f_2), (g_1, g_2))$, first note that A^{-1} is itself a contraction. Since A is linear and invertible, it suffices to prove that $||Ax||^2 > c||x||^2$ for all $x \in \mathbb{R}^2 - \{0\}$ and some $c > 1$ (since $\mathbb{R}^2$ is finite dimensional, it would actually suffice to prove that $c \geq 1$). It's easy enough to see that A's least eigenvalue is $15 - \sqrt{22}$, so we can take $c = 15 - \sqrt{22} > 1$.

Now we will prove that $\tilde{T}(f_1, f_2) = [T_1 f_1, T_2 f_2]^T$ is a contraction. For any $f_1, f_2, g_1, g_2 \in C_0(\mathbb{R})$, we have

$$\begin{aligned} d(\tilde{T}(f_1, f_2),&\tilde{T}(g_1, g_2)) \\ &\leq \sup_{x\in[0,1]} \int_0^1 |\sin(xt)||\sin(f_1(t)) - \sin(g_1(t))| \; dt \\ &+ \sup_{x\in[0,1]} \int_0^{1/2} |\cos(xt)||\cos(f_2(t)) - \cos(g_2(t))| \; dt. \end{aligned}$$

Using the bounds $|\sin(\theta)|, |\cos(\theta)| < 1$ and $|\sin'(\theta)|, |\cos'(\theta)| < 1$, we conclude that

$$\sup_{x\in[0,1]} \int_0^1 |\sin(xt)||\sin(f_1(t)) - \sin(g_1(t))| \; dt < \sup_{x\in[0,1]} |f_1(x) - g_1(x)| \text{ and}$$

$$\sup_{x\in[0,1]} \int_0^{1/2} |\cos(xt)||\cos(f_2(t)) - \cos(g_2(t))| \; dt < \sup_{x\in[0,1]} |f_2(x) - g_2(x)|.$$

Thus, $\tilde{T}$ is a contraction. It follows that T is the composition of contractions and hence is a contraction as well, so by the contraction mapping theorem, it has a fixed point (f_1, f_2), which solves the given equations.

Exercise 17. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set with finite Lebesgue measure $m(E)$. Define $f(r) = m((E + r) \cap E)$. Prove that f is continuous.

Solution 17. Problems like this come up a lot. There is a trick to them and like every problem with a trick, they are easiest when you know the trick. The trick is to approximate the function χ_E in L^1 (or any L^p, $p \in [1, \infty)$) with continuous functions f_n, prove that $Tf_n : r \mapsto \int \chi_{E+r}(x) f_n(x) \; dx$ is continuous for all n, then use Young's inequality to prove that T is continuous from L^1 to $C(\mathbb{R})$, so it sends the convergent sequence $f_n \to \chi_E$ to a convergent sequence in $C(\mathbb{R})$, and hence $T\chi_E$ is continuous. If that is clear to you, you can stop reading here, but I'll fill in more details and point out a connection to a large collection of problems in analysis in the next couple paragraphs.

If you want to calculate the measure of $(E+r)\cap E$ (or any set you don't know the measure of), one thing to try is to write it as the integral of a characteristic function, because we are often more comfortable manipulating integrals than measures. This is especially powerful when we want to find the measure of an intersection, because $\chi_{A\cap B} = \chi_A \chi_B$. Applying this, we see that

$$f(r) = \int \chi_{E+r}(x)\chi_E(x) \; dx = \int \chi_E(x - r)\chi_E(x) \; dx.$$

We could write this as an integral operator $T_K f(r) = \int K(x, r) f(x) \; dx$, where $K(x, r) = \chi_E(x - r)$. This problem asks you to prove a special case of a general principle that integral operators of this form should improve the regularity of the input function. In this case, we want to prove that T_K sends a function in $L^2(\mathbb{R})$ (we could choose other values of $p \in (1, \infty)$ here) to a function in $C(\mathbb{R})$, and we have the very useful property that $K(x, r) = F(r - x)$ for a function $F \in L^2(\mathbb{R})$. We will approach the remainder of the problem in this framework.

With our choice of K and F, we have that $T_K f(r) = F * f(r)$, so by Young's inequality

$$||T_K f||_{L^\infty} \leq ||F||_{L^2} ||f||_{L^2}.$$

Therefore T_K boundedly maps $L^2(\mathbb{R})$ into $L^\infty(\mathbb{R})$. Now, let's check that T_K sends $C_0(\mathbb{R}) \to C(\mathbb{R})$. Suppose $f \in C_0(\mathbb{R})$ and let E denote its support. Fix $\varepsilon > 0$. Since f is continuous and compactly supported, it is uniformly continuous, so we can find small δ such that if $|r-s| < \delta$, then $|f(r)-f(s)| < \varepsilon/(||F||_{L^2} m(E)^{1/2})$. Applying this, we see that if $|r-s| < \delta$, by Hölder's inequality

$$|T_K f(r) - T_K f(s)| < \int |F(x)||f(r-x)-f(s-x)| \, dx < ||F||_{L^2} m(E)^{1/2} \sup_{x\in\mathbb{R}} |f(r-x)-f(s-x)| < \varepsilon$$

We are almost done now. We know that T_K sends $C_0(\mathbb{R})$ to $C(\mathbb{R})$ and is bounded $L^2(\mathbb{R}) \to L^\infty(\mathbb{R})$. Now we want to prove that it sends $L^2(\mathbb{R}) \to C(\mathbb{R})$. Recall that $C_0(\mathbb{R})$ is dense in L^2. We also see that $C(\mathbb{R})$ is closed in L^∞. Since the L^∞ norm is the sup norm for continuous functions, a sequence of functions f_n in $C(\mathbb{R})$ converging to f in the L^∞ norm is Cauchy in the sup-norm, and hence has a continuous limit. But since limits are unique, that limit must be f, so $f \in C(\mathbb{R})$. Now if we have $f \in L^2(\mathbb{R})$, then we can take a sequence $f_n \in C_0(\mathbb{R})$ with $f_n \to f$ in L^2. Then $T_K(f_n) \to T_K f$ in L^∞ and each $T_K(f_n)$ is continuous, so $T_K f$ is continuous as well. We can take $K(x,r) = \chi_E(x-r)$ and $f(x) = \chi_E(x)$ to solve the original problem.

Exercise 18. Take $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^n$ and let $X + Y = \{x + y : x \in X, y \in Y\}$.

(1) Assume X is closed and Y is compact. Prove that $X + Y$ is closed.
(2) If Y is closed but not compact, is $X + Y$ closed? Prove or give a counterexample.

Solution 18.

(1) Suppose z is a limit of elements of $X+Y$, that is, $\lim_{n\to\infty} x_n + y_n = z$ for $x_n \in X$ and $y_n \in Y$. Since Y is compact, y_n has a convergent subsequence y_{n_k}, with limit $y \in Y$. Then $z = \lim_{k\to\infty} x_{n_k} + y_{n_k} = \lim_{k\to\infty} x_{n_k} + y$. It follows that x_{n_k} must be convergent as well, and since X is closed, it's limit x must fall in X. Then $z = x + y \in X + Y$, so $X + Y$ is closed.
(2) No. Take $X = \{-n : n \in \mathbb{N}\}$ and $Y = \{n + 1/n : n \in \mathbb{N}\}$. Both are discrete and hence closed, but neither are compact. Then $1/n \in X + Y$ for all $n \in \mathbb{N}$, but $0 \notin X + Y$.

Exercise 19. For $f \in L^2$, let $F(x) = \int_0^x f(t) \, dt$.

(1) Prove that
$$\int_0^1 \left(\frac{F(x)}{x}\right)^2 \, dx \le 4 \int_0^1 f^2(x) \, dx.$$
(2) For $x \in [0,1]$
$$Af(x) = \frac{1}{x\sqrt{1 + |\log(x)|}} \int_0^x f(t) \, dt.$$
Prove that if f_n is a sequence of continuous functions on $[0,1]$ with $\sup_n ||f_n||_{L^2([0,1])} \le 1$, then Af_n has a subsequence converging in the $L^2([0,1])$ norm.

Solution 19.

(1) Using Cauchy-Schwartz, we see that $|F^2(x)| \leq x \int_0^x f^2(t)\, dt$. Then $\frac{|F^2(x)|}{x} \int_0^x f^2(t)\, dt$. By dominated convergence, $\lim_{x\to 0} \int_0^x f^2(t)\, dt = 0$, so $\lim_{x\to 0^+} \frac{|F^2(x)|}{x} = 0$. Using this and integrating by parts, we see that

$$\int_0^1 \left(\frac{F(x)}{x}\right)^2 dx = \frac{F^2(x)}{x}\Big|_1^0 + 2\int_0^1 f(x)\frac{F(x)}{x}\, dx = -F^2(1) + 2\int_0^1 f(x)\frac{F(x)}{x}\, dx \leq 2\int_0^1 f(x)\frac{F(x)}{x}\, dx.$$

Then by Cauchy Schwartz, $\int_0^1 f(x)\frac{F(x)}{x}\, dx \leq \left(\int_0^1 f^2(x)\, dx\right)^{1/2} \left(\int_0^1 \frac{F(x)}{x}\, dx\right)^{1/2}$, so $\int_0^1 \left(\frac{F(x)}{x}\right)^2 dx \leq 2\left(\int_0^1 f^2(x)\, dx\right)^{1/2} \left(\int_0^1 \frac{F(x)}{x}\, dx\right)^{1/2}$. Rearranging and squaring, we arrive at the desired inequality.

(2) It would be great if we could apply Arzela-Ascoli directly to Af_n, but I don't think that is possible. But we can apply Arzela-Ascoli to $g_n(x) = \int_0^x f_n(t)\, dt$. To see that this is uniformly bounded, note that by Cauchy-Schwarz, $|g_n(x)| \leq x^{1/2}||f_n||_{L^2([0,1])} \leq 1$, since $x \in [0,1]$. We similarly have $|g_n(x) - g_n(y)| \leq |x-y|^{1/2}||f_n||_{L^2}$, so $\{g_n : n \in \mathbb{N}\}$ is equicontinuous. Then it has a uniformly convergent subsequence, g_{n_k} with limit g.

I first tried to prove that $Af_{n_k} \to \frac{g(x)}{x\sqrt{1+|\log(x)|}}$, but I didn't get anywhere doing that. I guessed we want to use part (a) somehow, which suggests that instead of comparing g_{n_k} to g (which might not be of the form $\int_0^x f(t)\, dt$), we should be comparing g_{n_k} to g_{n_j}. Conveniently, we know that g_{n_k} is a Cauchy sequence and, since L^2 is complete, it suffices to prove that Af_{n_k} is Cauchy.

Let's split up the integral to a part near 0 and everything else. Specifically,

$$||Af_{n_k} - Af_{n_j}||^2_{L^2([0,1])} = \int_0^\delta \frac{(g_{n_k}(x) - g_{n_j}(x))^2}{x^2(1+|\log(x)|)}\, dx + \int_\delta^1 \frac{(g_{n_k}(x) - g_{n_j}(x))}{x^2(1+|\log(x)|)}\, dx$$

By the previous part, $\int_0^\delta \frac{(g_{n_k}(x)-g_{n_j}(x))^2}{x^2(1+|\log(x)|)}\, dx \leq \frac{4}{1+|\log(\delta)|}\int_0^1 (f_{n_k}(x) - f_{n_j}(x))^2\, dx \leq \frac{C}{1+|\log(\delta)|}$ for some positive constant C, using the boundedness of $||f_n||_{L^2}$. On the other hand, $\int_\delta^1 \frac{(g_{n_k}(x)-g_{n_j}(x))^2}{x^2(1+|\log(x)|)}\, dx \leq \frac{1}{\delta^2}||g_{n_k} - g_{n_k}||^2_{L^\infty}$, using the fact that $\frac{1}{1+|\log(x)|} \leq 1$. Now to prove $||Af_{n_k} - Af_{n_j}||^2_{L^2([0,1])} < \varepsilon$, choose δ sufficiently small so that $\frac{1}{1+|\log(\delta)|} < \frac{\varepsilon}{2C}$ and then choose K sufficiently large so that for $k \geq K$, $||g_{n_k} - g_{n_k}||^2_{L^\infty} < \frac{\varepsilon\delta^2}{2}$. Combining these bounds, we see that for $k \geq K$, $||Af_{n_k} - Af_{n_j}||^2_{L^2([0,1])} \leq \varepsilon$. Since ε was arbitrary, we conclude that Af_{n_k} is Cauchy in the L^2 norm and hence Af_n has a convergent subsequence.

Exercise 20. Prove that there does not exist $f \in L^1(\mathbb{R})$ such that $g * f(x) = g(x)$ for any $g \in C_0(\mathbb{R})$ and $x \in \mathbb{R}$.

Solution 20. You can solve this easily using Fourier analysis and Riemann-Lebesgue theorem, but I will present a measure-theoretic proof. The equation $g*f = g$ has a solution if f is allowed to be a Radon measure $C_0^*(\mathbb{R})$. Let δ_0 the dirac mass at 0, so $\int_{-\infty}^\infty g(y-x)d\delta_0 x = g(y)$. If an L^1 function f also had that property, then $g * (f - \delta_0) = 0$ for any function $g \in C_0(\mathbb{R})$. It follows that $\langle g, f - \delta_0\rangle = 0$ ($\langle\cdot,\cdot\rangle$ denotes the dual pairing) for any $g \in C_0(\mathbb{R})$. But then $f - \delta_0 = 0 \in C_0^*(\mathbb{R})$, which implies that $f = \delta_0$. However, $\delta_0 \notin L^1(\mathbb{R})$, since it is not absolutely continuous with respect to the Lebesgue measure, so this is a contradiction.

Exercise 21. Let $I = [0,1)$ and given $N \in \mathbb{N}$ consider the dyadic intervals $I_{j,N} = [j2^{-N}, (j+1)2^{-N})$ for $j \in \{0, 1, \dots, 2^N - 1\}$. For a function $f \in L^1(I)$, define a sequence of function $E_N f : I \to \mathbb{R}$ by

$$E_N f(x) = 2^N \int_{I_{j,N}} f(t)\ dt \quad \text{ for } x \in I_{j,N}.$$

Show that $\lim_{N\to\infty} E_N f(x) = f(x)$ for a.e. $x \in I$.

Solution 21. We'll just run the usual proof of the Lebesgue differentiation theorem, with minor modifications to fit this case. Let $Mf(x) = \sup_{x\in I_{j,N}} 2^N \int_{I_{j,N}} |f|(t)\ dt$. Let's first prove that M enjoys the weak-type bound $||Mf||_{L^{1,\infty}} \leq C||f||_{L^1}$. We need to show that if $E_\delta = \{x \in \mathbb{R} : Mf(x) \geq \delta\}$, then $\delta m(E_\delta) \leq C||f||_{L^1}$ for an absolute constant C (that is, one that does not depend on f or δ). Now, for all $x \in E_\delta$, we can find an interval I_{j_x,N_x} containing x such that $\int_{I_{j_x,N_x}} |f|(x)\ dx \geq 2^{-N}\delta$. This gives a cover for E_δ. By the Vitali covering theorem, we can find a finite collection of disjoint sets $I_{j_1,N_1}, \dots, I_{j_m,N_m}$ such that if $\tilde{I}_{j,N}$ is the interval of length $\frac{5}{2^N}$ with the same center as $I_{j,N}$, then $\bigcup_{i=1}^m \tilde{I}_{j_i,N_i}$ covers E_δ, and hence $m(E_\delta) \leq 5\sum_{i=1}^m 2^{-N_i}$. Now we know

$$\int_I |f|(x)\ dx \geq \sum_{i=1}^m \int_{I_{j_i,m_i}} |f|(x)\ dx \geq \sum_{i=1}^m 2^{-N_i}\delta \geq \delta m(E_\delta)/5.$$

Since f and δ were arbitrary, we see that

$$||Mf||_{L^{1,\infty}} \leq 5||f||_{L^1},$$

as desired.

Now, let's complete the proof of the exercise. Define $E_\delta = \{x \in \mathbb{R} : \limsup_{N\to\infty} |E_N f(x) - f(y)| > \delta\}$. It suffices to show that $m(E_\delta) = 0$ for any $\delta > 0$, since then the set of points where $\lim_{N\to\infty} E_N f(x) \neq f(x)$ has measure 0. Let g be a continuous approximation of f, such that $||f - g||_{L^1} < \varepsilon$ for some small ε to be determined later. Then for any x and dyadic interval $I_{j,N}$ containing x,

$$2^N \int_{I_{j,N}} |f(t) - f(x)|\ dt \leq 2^N \int_{I_{j,N}} |f(t) - g(t)|\ dt + 2^N \int_{I_{j,N}} |g(t) - g(x)|\ dt + |g(x) - f(x)|.$$

By the bound on Mf proven in the previous paragraph, $\sup_{x\in I_{j,N}} 2^N \int_{I_{j,N}} |f(t) - g(t)|\ dt > \delta/3$ on a set of measure $< \frac{C||f-g||_{L^1}}{\delta}$. By Markov's inequality (which says $||f||_{L^{1,\infty}} \leq ||f||_{L^1}$ and follows very easily from the layer-cake formula), $|g(x) - f(x)| > \delta/3$ on a set of measure $< \frac{3||f-g||^{L^1}}{\delta}$. And since g is continuous, $\limsup_{N\to\infty} 2^N \int_{I_{j,N}} |g(t) - g(x)|\ dt = 0$. It follows that $\limsup_{N\to\infty} |E_N f(x) - f(y)| \geq \delta$ on a set of measure $< \frac{C||f-g||_{L^1}}{\delta} < \frac{C\varepsilon}{\delta}$. Taking ε arbitrarily small, we conclude that $m(E_\delta) = 0$. Since δ was arbitrary, we see that $\lim_{N\to\infty} |E_N f(x) - f(x)| = 0$ almost everywhere.

Exercise 22. Suppose f, f_n are Lebesgue measureable functions on $[0,1]$ finite a.e.. Show that $f_n \to f$ in measure if and only if

$$\lim_{n\to\infty} \int_0^1 \frac{|f_n - f|(x)}{1 + |f_n - f|(x)}\ dx = 0.$$

Solution 22. Suppose $f_n \to f$ in measure. Let f_{n_k} be a subsequence of f_n. Then it also converges in measure to f and hence has a subsequence $f_{n_{k_j}}$ converging pointwise almost everywhere. Note that $\frac{|f_{n_{k_j}}-f|(x)}{1+|f_{n_{k_j}}-f|(x)} \in [0,1]$, so by dominated converge,ce, $\lim_{j\to\infty} \int_0^1 \frac{|f_{n_{k_j}}-f|(x)}{1+|f_{n_{k_j}}-f|(x)}\, dx = \int_0^1 \lim_{j\to\infty} \frac{|f_{n_{k_j}}-f|(x)}{1+|f_{n_{k_j}}-f|(x)}\, dx = 0$. Since every subsequence of $\{\int_0^1 \frac{|f_n-f|(x)}{1+|f_n-f|(x)}\, dx\}$ has a subsubsequence converging to 0, we see that $\lim_{n\to\infty} \int_0^1 \frac{|f_n-f|(x)}{1+|f_n-f|(x)}\, dx = 0$.

Now suppose $\lim_{n\to\infty} \int_0^1 \frac{|f_n-f|(x)}{1+|f_n-f|(x)}\, dx = 0$. Fix $\varepsilon > 0$ and let $E_n = \{x \in [0,1] : |f_n - f|(x) > \varepsilon\}$. We need to show that $m(E_n) \to 0$. Note that for $x \in E_n$, $\frac{|f_n-f|(x)}{1+|f_n-f|(x)} = 1 - \frac{1}{1+|f_n-f|(x)} \geq 1 - \frac{1}{1+\varepsilon} = \frac{\varepsilon}{1+\varepsilon}$. Then $\int_0^1 \frac{|f_n-f|(x)}{1+|f_n-f|(x)}\, dx \geq E_n \frac{\varepsilon}{1+\varepsilon}$, so $m(E_n) \leq \frac{1+\varepsilon}{\varepsilon} \int_0^1 \frac{|f_n-f|(x)}{1+|f_n-f|(x)}$, and so since $\lim_{n\to\infty} \int_0^1 \frac{|f_n-f|(x)}{1+|f_n-f|(x)} = 0$, $\lim_{n\to\infty} m(E_n) = 0$ as well. Since $\varepsilon > 0$ was arbitrary, it follows that $f_n \to f$ in measure.

Exercise 23. For $f, g \in L^2[0,1]$, let $\langle f, g\rangle = \int_0^1 f(x)\overline{g}(x)\, dx$ and set

$$g_n(x) = \frac{n^{2/3}\sin(n/x)}{xn+1}.$$

Does there exists $\alpha > 0$ such that

$$\sum_{n=1}^{\infty} |\langle f, g_n\rangle|^{\alpha} < \infty.$$

hold for every $f \in L^2$? *Hint: is* $||g_n||_{L^2}$ *a bounded sequence?*

Solution 23. The first thing you should think to do is answer the question asked in the hint. The answer it that $||g_n||_{L^2}$ is not bounded. It suffices to prove that $||g_n||_{L^2}^2 = \int_0^1 \frac{n^{4/3}|\sin(n/x)|^2}{(nx+1)^2}\, dx$ is unbounded. The $n^{4/3}$ is an obvious source of growth for this integral. On the other hand, we should expect that the $|\sin(n/x)|^2$ term will contribute less as n gets larger, essentially because the periods of $\sin(n/x)$ get narrower as n get's larger. We need to quantify the rate of decay to ensure that it is slower than the rate of growth, and hence see that $||g_n||_{L^2}^2$ is unbounded.

It suffices to prove that $\int_0^1 \frac{|\sin(n/x)|^2}{(nx+1)^2}\, dx \geq \frac{C}{n}$ for some fixed constant C. Keeping track of constants in this would be a big pain though, so I'll use asymptotic notation instead. To make the denominator go away, we will limit the domain of the integral to $[0, 1/n]$, on when $(nx+1)^2 \in [1,4]$. We therefore reduce our problem to proving $\int_0^{1/n} |\sin(n/x)|^2\, dx \gtrsim \frac{1}{n}$. Now substitute $u = n/x$. Then $-\frac{n}{u^2}\, du = dx$, so

$$\int_0^{1/n} |\sin(n/x)|^2\, dx = n\int_{n^2}^{\infty} \frac{|\sin(u)|^2}{u^2}\, du.$$

We can bound this below by restricting the domain of integration to where $\sin(u) \geq \frac{1}{2}$, which is contained in $I = \bigcup_{k\in\mathbb{Z}\cap[n^2,\infty)} I_k$, where $I_k = [k\pi - \pi/3, k\pi + \pi/3]$. Then

$$n\int_{n^2}^{\infty} \frac{|\sin(u)|^2}{u^2}\, du \gtrsim n \sum_{k\in\mathbb{Z}\cap[n^2,\infty)} \int_{I_k} \frac{1}{u^2}\, du.$$

On I_k, $\frac{1}{u^2} \approx \frac{1}{k^2}$. Each I_k has constant length, so

$$n \sum_{k=n^2}^{\infty} \int_{I_k} \frac{1}{u^2}\, du \gtrsim n \sum_{k=n^2}^{\infty} \frac{1}{k^2}.$$

By the integral comparison test, $\sum_{k=n^2}^{\infty} \frac{1}{k^2} \approx \frac{1}{n^2}$, so $n \sum_{k=n^2}^{\infty} \frac{1}{k^2} \approx \frac{1}{n}$. Putting this all together, we see that $\int_0^1 \frac{|\sin(n/x)|^2}{(nx+1)^2}\, dx \gtrsim \frac{1}{n}$, as desired. Hence, $||g_n||_{L^2}^2$ is unbounded.

Now that we are done sorting out the hint, we should figure out why the author added the hint. In other words, what is the relation between $||g_n||_{L^2}$ being unbounded and $\sum_{n=1}^{\infty} |\langle f, g_n\rangle|^\alpha$ being infinite. It is reasonable to guess that if $||g_n||_{L^2}$ is not small, then $\sum_{n=1}^{\infty} |\langle f, g_n\rangle|^\alpha$ is not small (that is, finite) either. Turning that into an actual proof requires a bit of functional analysis.

First, note that if there exists $\alpha > 0$ such that $\sum_{n=1}^{\infty} |\langle f, g_n\rangle|^\alpha < \infty$, then by the divergence test (who knew 221 content was important for the qual!), $\lim_{n\to\infty} |\langle f, g_n\rangle|^\alpha = 0$. It follows that $\lim_{n\to\infty} |\langle f, g_n\rangle| = 0$ and, since $f \in L^2$ in arbitrary, we see that g_n converges weakly 0 in L^2. But by the open mapping theorem, we know weakly convergent sequences must be bounded in norm, and hence $||g_n||_{L^2}$ would be bounded, contradicting our earlier proof otherwise. Hence, $\sum_{n=1}^{\infty} |\langle f, g_n\rangle|^\alpha = \infty$ for any $\alpha > 0$.

Exercise 24. Prove that there is a distribution $u \in \mathcal{D}'(\mathbb{R})$ so that its restriction to $(0, \infty)$ is given by

$$\langle u, f\rangle = \int_0^\infty x^{-2} \cos(x^{-2}) f(x)\, dx$$

for all $f \in C^\infty(\mathbb{R})$ compactly supported on $(0, \infty)$ and $\langle u, f\rangle = 0$ for all $f \in C^\infty(\mathbb{R})$ compactly supported on $(-\infty, 0)$.

Solution 24. One thing to try on problems like this is to use integration by parts to make the term that is bad at 0 less bad, at the cost of putting derivatives on f (which isn't really a cost at all when we are talking about distributions) and having boundary terms behave poorly at 0 (but we only care about f supported away from 0, so this won't be an issue).

In this case, this procedure works pretty efficiently, although it still took me a couple tries to get everything working. The first thing I tried was integrating x^{-2} and differentiating $\cos(x^{-2})f(x)$, which didn't immediately work because the derivatives of $\cos(x^{-2})$ created singularities at 0 as well. But doing that made me realize that $x^{-2}\cos(x^{-2})$ very nearly has an elementary antiderivative. Since $\frac{d}{dx}\sin(x^{-2}) = -2x^{-3}\cos(x^{-2})$, we will write $\int_0^\infty x^{-2}\cos(x^{-2})f(x)\, dx = \int_0^\infty x^{-3}\cos(x^{-2})(xf(x))\, dx$. Integrating by parts, this becomes $\frac{\sin(x^{-2})f(x)}{2}\Big|_\infty^0 + \int_0^\infty \frac{\sin(x^{-2})}{2}(f'(x) + xf(x))\, dx$. Since $\sin(x^{-2})$ and $x\sin(x^{-2})$ are in L^1_{loc}, the map $\langle u, f\rangle = \int_0^\infty \frac{\sin(x^{-2})}{2}(f'(x) + xf(x))\, dx$ is a well-defined distribution. It certainly sends functions supported on $(-\infty, 0)$ to 0, and undoing the integration by parts I started with, we see that equals $\int_0^\infty x^{-2}\cos(x^{-2})f(x)\, dx$ when f is compactly supported on $(0, \infty)$.

Exercise 25. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{x}{n}\right)$$

.

(1) Does it converge uniformly on $[0, 1]$?
(2) Does it converge uniformly on $[0, \infty)$?

Solution 25.

(1) It does converge uniformly in the given range. It suffices to prove that for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $M \geq N$, then $\left|\sum_{n=M}^{\infty} \frac{1}{n} \sin\left(\frac{x}{n}\right)\right| \leq \varepsilon$. Note that by the standard result that $|\sin(x)| \leq |x|$,

$$\left|\sum_{n=N}^{\infty} \frac{1}{n} \sin\left(\frac{x}{n}\right)\right| \leq \sum_{n=N}^{\infty} \frac{1}{n}\left|\sin\left(\frac{x}{n}\right)\right| \leq \sum_{n=N}^{\infty} \frac{x}{n^2} \leq \sum_{n=N}^{\infty} \frac{1}{n^2}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, we know that $\lim_{N\to\infty} \sum_{n=N}^{\infty} \frac{1}{n^2} = 0$, and hence for N sufficiently large and all $M \geq N$, $\sum_{n=M}^{\infty} \frac{1}{n^2} < \varepsilon$. For these values of M, it follows that $\left|\sum_{n=M}^{\infty} \frac{1}{n} \sin\left(\frac{x}{n}\right)\right| \leq \varepsilon$ as well, so we have uniformy convergence.

(2) Suppose the series converged uniformly on $[0, \infty)$. Then there exists N sufficiently large so that for all $M \geq N$ and all x, $\left|\sum_{n=M}^{\infty} \frac{1}{n} \sin\left(\frac{x}{n}\right)\right| < 1/100$. It follows that $\left|\sum_{n=N}^{\infty} \frac{1}{n} \sin\left(\frac{N}{n}\right)\right| < 1/100$. But $\frac{N}{n} \in [0,1]$ for all $n \geq N$, so $\sin\left(\frac{N}{n}\right) \geq \frac{N}{10n}$. It follow that $\sum_{n=N}^{\infty} \frac{1}{n} \sin\left(\frac{N}{n}\right) \geq \frac{N}{10} \sum_{n=N}^{\infty} \frac{1}{n^2}$. By the integral test, $\sum_{n=N}^{\infty} \frac{1}{n^2} \geq \frac{1}{N}$, so $\frac{N}{10} \sum_{n=N}^{\infty} \frac{1}{n^2} \geq \frac{1}{10}$, contradicting our assumption that it was less than $\frac{1}{100}$. Hence, the series does not converge uniformly.

Exercise 26. Can one find a bounded sequence of real numbers $x_n, n \in \mathbb{Z}$ that satisfies

$$x_n = \sin(n) + 0.5x_{n-1} + 0.4\sin(x_{n+1})$$

for every $n \in \mathbb{Z}$?

Solution 26. This looks slightly different than other contraction mapping problems, but it is one. We need a metric to apply the contraction mapping theorem. The simplest metrics on sequence spaces are given by ℓ^p norms. In this case, our operator will map the sequence $x_n \equiv 0$ to the sequence $x_n = \sin(n)$, so it will not map into an ℓ^p space other than ℓ^∞. We will proceed using the ℓ^∞ metric.

Let $\ell^\infty(\mathbb{Z})$ be the space of integer indexed sequences of real numbers such that the norm $||(x_n)_{n\in\mathbb{N}}||_{\ell^\infty(\mathbb{Z})} := \sup_{n\in\mathbb{Z}} |x_n|$ is finite. Define the operator $T : \ell^\infty \to \ell^\infty$ by $T((x_n)_{n\in\mathbb{N}})_m = \sin(m) + 0.5x_{m-1} + 0.4\sin(x_{m+1})$. First, note that this is a well-defined mapping (that is, $T((x_n)_{n\in\mathbb{N}}) \in \ell^\infty(\mathbb{Z})$ for all $(x_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{Z})$), since

$$||T((x_n)_{n\in\mathbb{N}})||_{\ell^\infty(\mathbb{Z})} \leq 1.5 + 0.5||(x_n)_{n\in\mathbb{N}}||_{\ell^\infty(\mathbb{Z})} < \infty$$

To see that it is a contraction, note that

$$\begin{aligned} |(T((x_n)_{n\in\mathbb{N}}) - T((y_n)_{n\in\mathbb{N}}))_m| &\leq 0.5|x_{m-1} - y_{m-1}| + 0.4|\sin(x_{m+1}) - \sin(y_{m+1})| \\ &\leq 0.5|x_{m-1} - y_{m-1}| + 0.4|x_{m+1} - y_{m+1}| \\ &\leq 0.9||x - y||_{\ell^\infty(\mathbb{Z})}. \end{aligned}$$

Since m was arbitrary, we conclude that $||T((x_n)_{n\in\mathbb{N}}) - T((y_n)_{n\in\mathbb{N}})||_{\ell^\infty(\mathbb{Z})} \leq 0.9||(x_n)_{n\in\mathbb{N}} - (y_n)_{n\in\mathbb{N}}||_{\ell^\infty(\mathbb{Z})}$, so T is a contraction mapping. It follows that it has a fixed point $(x_n)_{n\in\mathbb{N}}$, which hence satisfies $x_n = \sin(n) + 0.5x_{n-1} + 0.4\sin(x_{n+1})$ for all n.

Exercise 27. Suppose S is the set of real-valued functions continuous g on $[0,1]$ that satisfy two conditions:

$$\left|\int_0^1 g(x)\ dx\right| \le 1$$

and

$$|g(x) - g(y)| \le |x-y|^{1/2}$$

for each $x, y \in [0,1]$. Consider the functional

$$F(g) = \int_0^1 (1-5x^2)g^{10}(x)\ dx.$$

Is F bounded on S? Does it acheive it's maximum on S?

Solution 27. First, let's prove that F is bounded. The conditions on S do not individually imply that each g is bounded, but taken together, they imply that each $g \in S$ is bounded. Suppose that $g(x) > 2$ for some $x \in [0,1]$. Then since $|g(y) - g(x)| < 1$ for all $y \in [0,1]$, we have that $g(y) > 1$ for all $y \in [0,1]$, in which case $\int_0^1 g(y)\ dy > 1$, a contradiction. Similarly, we cannot have $g(x) < -2$. Then $|g(x)| < 2$ for all $x \in [0,1]$, so $|g^{10}(x)| < 2^{10}$ for all $x \in [0,1]$.

Now, apply Hölder's inequality to see that

$$|F(g)| \le \int_0^1 (1-5x^2)\ dx \sup_{x\in[0,1]} |g^{10}(x)| < C$$

for some fixed constant C not depending on g.

To see that it achieves its maximum is somewhat trickier. Suppose $M = \sup_{g\in S} F(g)$. Then there is a sequence $g_n \in S$ such that $\lim_{n\to\infty} F(g_n) = M$. Let's prove that g_n subsequentially converges to some $g \in S$. Since F is bounded, it is continuous on S, and hence g will achieve the maximum.

We will prove g_n subsequentially converges using Arzela-Ascoli. We have already noted that elements of S are uniformly bounded. The second condition on elements of S also implies that S is equicontinuous. If we take $\varepsilon > 0$, then if $|x-y| < \varepsilon^2$, then for any $g \in S$, $|g(x) - g(y)| < \varepsilon$. It follows that S is an equicontinuous family, so g_n has a convergent subsequence g_{n_k}, converging to some g. We have already noted that the F is continuous, so $F(g) = M$. It remains to prove that $g \in S$. Since g_n is uniformly bounded, we can apply dominated convergence to see that $\lim_{k\to\infty} \int_0^1 g_{n_k}(x)\ dx = \int_0^1 g(x)\ dx$. Hence, $\left|\int_0^1 g(x)\ dx\right| \le 1$. For other condition, we have that $|g(x) - g(y)| = \lim_{k\to\infty} |g_n(x) - g_n(y)| \le \sqrt{|x-y|}$ for any pair x, y. Hence, $g \in S$ and F achieves its maximum on S.

Exercise 28. Let $f_n : X \to \overline{\mathbb{R}}$ be a sequence of measurable functions on a finite measure space X, so that $|f_n(x)| < \infty$ for almost every $x \in X$. Show that there is a sequence A_n of positive real numbers so that

$$\lim_{n\to\infty} \frac{f_n(x)}{A_n} = 0$$

almost everywhere. *Hint: Borel-Cantelli.*

Solution 28. The Borel-Cantelli lemma states that if E_n is a sequence of measurable sets and $\sum_{n\in\mathbb{N}} m(E_n) < \infty$, then $\limsup E_n$ has measure 0. It would be rather convenient if we could choose our sets E_n such that if $x \notin \limsup E_n$ (that is, $x \notin E_n$ for all but finitely many n), then $\lim_{n\to\infty} \frac{f_n(x)}{A_n} = 0$. We then could choose E_n to be the set of xs where $\frac{f_n(x)}{A_n}$ decays slower than a certain rate (say, $\frac{f_n(x)}{A_n} > 1/n$). We know that outside of $\limsup E_n$, $\frac{f_n(x)}{A_n} \leq 1/n$ for all n sufficiently large, and hence $\lim_{n\to\infty} \frac{f_n(x)}{A_n} = 0$. It then suffices to prove that $\sum_{n\in\mathbb{N}} m(E_n) < \infty$, and hence $\limsup E_n = 0$. We will give an explicit construction the E_n, then prove that $\sum_{n\in\mathbb{N}} m(E_n) < \infty$, completing the problem

Since f_n is finite a.e. and X has finite measure, $\lim_{c\to\infty} m(\{x : f_n(x) > c\}) = 0$. It follows that we can choose A_n sufficiently large so that $m(\{x : f_n(x) > A_n/n\}) < 2^{-n}$. Let $E_n = \{x : f_n(x) > A_n\}$. Since f_n is a measurable function, E_n is a measurable set. Additionally, $\sum_{n\in\mathbb{N}} m(E_n) < 1$. As discussed previously, Borel-Cantelli implies that $\limsup E_n$ has measure 0 and outside of $\limsup E_n$, $\lim_{n\to\infty} \frac{f_n(x)}{A_n} = 0$, completing the problem.

Exercise 29. Let $I = [a, b]$ and let $L^2(I)$ be the space of square-integrable functions on $[a, b]$ with scalar product $\langle f, g\rangle = \int_a^b f(t)\overline{g}(t)\, dt$. Let $p_0, p_1, \ldots$ be a sequence of real-valued polynomials p_n of degree exactly n such that

$$\int_a^b p_j(x)p_k(x)\, dx = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}.$$

Prove that $\{p_n\}_{n=0}^{\infty}$ is a *complete* orthonormal system.

Solution 29. A complete orthonormal system is an orthonormal system with dense span. It is immediate from the given definition that $\{p_n\}_{n\in\mathbb{N}}$ forms an orthonormal system, it suffices to show that it is complete. In other words, we need to prove that any $f \in L^2(I)$ can be approximated with polynomials. We have a very theorem about polynomial approximation called the Stone-Weierstrass theorem, which states that any continuous functions on $[a, b]$ can be approximated by polynomials in the $L^\infty([a, b])$ norm. The $L^2([a, b])$ norm is less than $\sqrt{b-a}$ times the $L^\infty([a, b])$ norm, so it follows that any continuous function on $[a, b]$ can be approximated in the $L^2([a, b])$ norm by polynomials as well. We then see that $\text{span}((p_n)_{n\in\mathbb{N}})$ is dense in $C([a, b])$ equipped with the $L^2([a, b])$ norm.

It remains to prove that $C([a, b])$ is dense in $L^2([a, b])$. This is a pretty standard result, but for completeness, I will prove this (or at least, reduce to very results I very much doubt you will be expected to prove). First, step functions are dense in $L^2([a, b])$, so it suffices to prove that $C([a, b])$ is dense in the set of $L^2([a, b])$ step functions. Since step functions are linear combinations of simple functions, it suffices to prove that $C([a, b])$ is dense the in the space of simple functions, in other words, that χ_E is a limit of continuous functions for any $E \in [a, b]$. By the regularity of the Lebesgue measure, for any $\varepsilon > 0$, we can find a compact set K and an open set U such that $K \subset E \subset U$ and $m(U \setminus K) < \varepsilon^2$. By Urysohn's lemma, we can find a continuous function $f : [a, b] \to [0, 1]$ equal to 1 on K and 0 on U. Then $|f - \chi_E|$ is at most 1 and is only non-zero on $U \setminus K$. Thus, $||f - \chi_E||_{L^2([a,b])} < \varepsilon$. Since ε was arbitrary, we are done.

Exercise 30. For $x, y \in \mathbb{R}$, let $K(y) = \pi^{-1}(1 + y^2)^{-1}$, and for $t > 0$ let

$$P_t f(x) = \int_{-\infty}^{\infty} t^{-1}K(t^{-1}y)f(x - y)\, dy.$$

(1) Show that if f is continuous and compactly supported, then

$$\lim_{t\to 0^+}\sup_{x\in\mathbb{R}}|P_tf(x)-f(x)|=0.$$

(2) Let $p\geq 1$. For $f\in L^p(\mathbb{R})$ denote by Mf the Hardy-Littlewood maximal function of f. Show that there is a constant $C>0$ so that for all $f\in L^p(\mathbb{R})$ the inequality

$$|P_tf(x)|\leq CMf(x)$$

holds for every $x\in\mathbb{R}$ and every $t>0$.

(3) If $f\in L^1(\mathbb{R})$, prove that $\lim_{t\to 0+}P_tf(x)=f(x)$ for almost every $x\in\mathbb{R}$.

Solution 30.

(1) Fix $\varepsilon>0$, let's prove that for t sufficiently large and all $x\in\mathbb{R}$, $|P_t(x)-f(x)|<\varepsilon$. Since f is continuous and compactly supported, $M:=\sup_{x\in\mathbb{R}}|f(x)|<\infty$ and there exists $\delta>0$ such that if $|h|<\delta$, then $|f(x+h)-f(x)|<\frac{\varepsilon}{2}$. Note that $\int K(y)\,dy=\frac{\arctan(x)}{\pi}+C$, so $\int_{-\infty}^{\infty}K(y)\,dy=1$, and by change of variables, $\int_{-\infty}^{\infty}t^{-1}K(t^{-1}y)\,dy=1$. We also know that by dominated convergence that $\lim_{t\to 0^+}\int_{|y|\geq\delta/t}K(y)\,dy=0$, so for t sufficiently large, $\int_{|y|\geq\delta/t}K(y)\,dy\geq\frac{\varepsilon}{2M}$.

Then for any $x\in\mathbb{R}$,

$$\begin{aligned}
|P_tf(x)-f(x)| &= \left|\int_{-\infty}^{\infty}t^{-1}K(t^{-1}y)[f(x-y)-f(x)]\,dy\right|\\
&\leq \left|\int_{-\infty}^{\infty}K(y)[f(x-ty)-f(x)]\,dy\right|\\
&\leq \int_{|y|<\delta/t}K(y)|f(x-ty)-f(x)|\,dy+\int_{|y|\geq\delta/t}K(y)|f(x-ty)-f(x)|\,dy\\
&\leq \int_{|y|<\delta/t}K(y)\frac{\varepsilon}{2}\,dy+\int_{|y|\geq\delta/t}K(y)M\,dy\\
&\leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
\end{aligned}$$

Since $\varepsilon>0, x\in\mathbb{R}$ were arbitrary, $\lim_{t\to 0^+}\sup_{x\in\mathbb{R}}|P_tf(x)-f(x)|=0$.

(2) Recall that the Hardy-Littlewood maximal function is defined to be

$$Mf(x)=\sup_{r>0}\frac{1}{2r}\int_{-r}^{r}|f|(x-y)\,dy.$$

The idea here is to approximate the kernel $K(y/t)$ from above with integrals of f over intervals, then bound those from above with the Hardy-Littlewood maximal function.

For $n\in\mathbb{N}$ and $t>0$, define $I_{n,t}=\{x:K(x/t)\geq 2^{-n}]\}$ and define $I_{0,t}=\emptyset$. Since K increases for negative x and decreases for positive x, $I_{n,t}$ are intervals, since K is bounded above by 1, $\mathbb{R}=\bigcup_{n\in\mathbb{N}}I_{n,t}$, and I_n is an increasing family: $I_1\subset I_2\subset\ldots$. We can compute that if $y\in I_{n,t}$, then $1+(y/t)^2\leq 2^n$, so $y^2\leq t^22^n$, and hence

$|y| \le t2^{n/2}$, and hence $I_{n,t} \subset [-t2^{n/2}, t2^{n/2}]$. It follows that

$$\begin{aligned} P_t f(x) &\le \sum_{n=1}^{\infty} \frac{1}{t} \int_{I_{n,t} \setminus I_{n-1,t}} 2^{1-n} |f|(x-y)\, dy \\ &\le \sum_{n=1}^{\infty} \frac{2}{2^n t} \int_{I_{n,t}} |f|(x-y)\, dy \\ &\le \sum_{n=1}^{\infty} \frac{2}{2^n t} \int_{-t2^{n/2}}^{t2^{n/2}} |f|(x-y)\, dy \\ &\le \sum_{n=1}^{\infty} \frac{42^{n/2} t}{2^n t} Mf(x) \\ &\le 4 \frac{\sqrt{2}}{\sqrt{2}-1} Mf(x). \end{aligned}$$

Where the last inequality is by summing the geometric series $\sum_{n=0}^{\infty} \frac{1}{\sqrt{2}^n}$. We will take the latter constant as C, note that it does not depend on t or x, so this reasoning completes the problem.

(3) I assume we can take the weak-L^1 boundedness of the Hardy-Littlewood maximal function for granted, in which case we can conclude from the previous part that $\tilde{M}f(x) = \sup_{t>0} P_t|f|(x)$ is weak-L^1 bounded as well, that is $||Mf||_{L^{1,\infty}} \le ||f||_{L^1}$. From here, we will reproduce the end of the proof of the Lebesgue differentiation theorem.

Fix $m \in \mathbb{N}$ and let $E_m = \{x \in \mathbb{R} : \limsup_{t\to 0^+} |P_t f(x) - f(x)| \ge 1/m\}$. For $x \notin \bigcup_{m\in\mathbb{N}} E_m$, $\limsup_{t\to 0^+} |P_t f(x) - f(x)| = 0$, so $\lim_{t\to 0} P_t f(x) = f(x)$. We will conclude by proving that $|E_m| = 0$ for all m, so $\left|\bigcup_{m\in\mathbb{N}} E_m\right| = 0$. Fix $m \in \mathbb{N}$ and $\varepsilon > 0$, let's prove that $|E_m| < \varepsilon$. Since continuous, compactly supported functions are dense in $L^1(\mathbb{R})$, we can find $g \in C_c(\mathbb{R})$ such that $||g - f||_{L^1}$ is a small value to be determined later. It follows that g differs from f by more than $\frac{1}{3m}$ on a set of F of measure $\le 3m||g - f||_{L^1}$. We know for all $x \in \mathbb{R}$, $|P_t f(x) - f(x)| \le |P_t f(x) - P_t g(x)| + |P_t g(x) - g(x)| + |g(x) - f(x)|$. If $x \in F^c$, then $|g(x) - f(x)| < \frac{1}{3m}$. We know by the first problem that $\limsup_{t\to 0} |P_t g(x) - g(x)| = 0$. By definition, $|P_t f(x) - P_t g(x)| \le \tilde{M}(f-g)(x)$, so by the weak-$L^1$ bound previously discussed, $|P_t f(x) - P_t g(x)| \le \frac{C}{3m}$ outside of a set G of measure $\le 3m||f - g||_{L^1}$. Then for $x \notin F \cup G$, $\limsup_{t\to 0^+} |P_t f(x) - f(x)| \le \frac{2}{3m}$, so $x \notin E_m$. Since $E_m^c \supset (F \cup G)^c$, we have that $E_m \subset F \cup G$, so $|E_m| < |F| + |G| \le m(3 + 3C)||f - g||_{L^1}$. Choose $||f - g||_{L^1} \le \frac{\varepsilon}{m(3+3C)}$, and we see that $|E_m| < \varepsilon$. Since ε was arbitrary, we have $|E_m| = 0$, as desired.

Exercise 31. On $\mathbb{R} \setminus \{0\}$ define $f(x) = |x|^{-7/2}$. Find a tempered distribution $h \in \mathcal{S}'(\mathbb{R})$ so that $f = h$ on $\mathbb{R} \setminus \{0\}$.

Solution 31. We will integrate by parts. Suppose $\varphi \in \mathcal{S}(R)$ and is compactly supported away from 0. Then integrating by parts (I'll leave it to the reader to do this carefully - you want to break up the domain into $(-\infty, 0)$ and $(0, \infty)$, and then use the fact that φ is supported away from 0 to ensure the boundary terms vanish)

$$\int |x|^{-7/2}\varphi(x)\;dx = \frac{2}{5}\int |x|^{-5/2}\varphi'(x)\;dx = \frac{4}{15}\int |x|^{-3/2}\varphi''(x)\;dx = \frac{8}{15}\int |x|^{-1/2}\varphi^{(3)}(x)\;dx.$$

The final term is a well-defined tempered distribution: it is in $L^1((-1,1))$ and has (much better than) polynomial growth as $|x| \to \infty$. So we will take that to be h, and undoing the integration by parts written out above, we see that $h = f$ on $\mathbb{R} \setminus \{0\}$, as desired.

Exercise 32. Let X, Y be σ-finite measure space with positive measures $d\mu, d\nu$ respectively and let K be a measurable function on $X \times Y$. Let u and w be nonnegative measurable functions on X, Y respectively. Assume that $u(x) > 0$ and $w(y) > 0$ a.e.. Suppose $1 < p < \infty$

$$\int_X |K(x,y)|u(x)d\mu(x) \le Bw(y), \quad \nu - \text{a.e.}$$

$$\int_Y |K(x,y)|w(y)^{1/(p-1)}d\nu(y) \le Au(x)^{1/(p-1)} \quad \mu - \text{a.e.}$$

Let T be defined by $Tf(x) = \int_Y K(x,y)f(y)\;d\nu(y)$. Show that T maps $L^p(Y)$ to $L^p(X)$ with operator norm bounded by $A^{1-1/p}B^{1/p}$.

Solution 32. This is a tricky one and requires some functional analysis techniques that are not commonly discussed. Let's start with some simplifying assumptions: assume $X = Y = \mathbb{R}$, μ and ν are simply the Lebesgue measure, and $u \equiv w \equiv 1$. In hindsight, the first two assumptions will only provide moral support, as we will never use any properties unique to specific measure spaces or measures. The final assumption is more significant, but still will be easy enough to relax when the time comes. With these assumptions, are question is now to prove that if $\int_{\mathbb{R}} |K(x,y)|\;dx \le B$ for almost every y and $\int_{\mathbb{R}} |K(x,y)|\;dy \le A$ for almost every x, then if $Tf(x) = \int_{\mathbb{R}} K(x,y)f(y)\;dy$ satisfies $||Tf||_{L^p} \le A^{1/p'}B^{1/p}||f||_{L^p}$, where p' is the Hölder conjugate of p. This feels more comfortable for me and hopefully for you as well.

A useful tool for proving L^p bounds of an integral operator is the principle of duality. For any $1 < p < \infty$, let p' be the Hölder conjugate of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$. Then

$$||Tf||_{L^p} = \sup_{g \neq 0 \in L^{p'}} \frac{\langle Tf, g\rangle}{||g||_{L^{p'}}} = \sup_{g \neq 0 \in L^{p'}} \frac{\int_{\mathbb{R}\times\mathbb{R}} K(x,y)f(y)g(x)\;dy\;dx}{||g||_{L^{p'}}}.$$

I think you are free to use this without proof, as it is somewhat challenging to prove. Doing this removes the strange asymmetry of having bounds in the integral of K over each individual variable, but an operator that only integrates over one variable. Now we see that it suffices to prove something much more symmetric:

$$\int_{\mathbb{R}\times\mathbb{R}} K(x,y)f(y)g(x)\;dy\;dx \le A^{1/p'}B^{1/p}||f||_{L^p}||g||_{L^{p'}}.$$

It is easy to see that we only need to consider the case when K, f, and g are all non-negative, as the general case would follow from the fact that $\int_{\mathbb{R}\times\mathbb{R}} K(x,y)f(y)g(x)\;dy\;dx \le \int_{\mathbb{R}\times\mathbb{R}} |K(x,y)||f(y)|g(x)|\;dy\;dx$.

Now, we will do something more unusual. Recall that Hölder's inequality holds for any measure space and non-negative measure, not just the Lebesgue measure. In particular, it holds for the measure $|K(x,y)|\ dx\ dy$. Applying this, we see that

$$\int_{\mathbb{R}\times\mathbb{R}} |K(x,y)||f(y)|g(x)|\ dy\ dx \leq \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |K(x,y)||f(y)|^p\ dx\ dy\right)^{1/p}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} |K(x,y)||g(y)|^{p'}\ dx\ dy\right)^{1/p'}$$

For the first integral, we will first integrate in x (since everything is non-negative, we are free to integrate in whatever order feels appropriate). Using the given bound $\int |K(x,y)|\ dx \leq B$, we see that the first integral is bounded above by $B^{1/p}||f||_{L^p}$. Similarly, the second integral is bounded above by $A^{1/p'}||g||_{L^{p'}}$. All together, $\int_{\mathbb{R}\times\mathbb{R}} |K(x,y)||f(y)|g(x)|\ dy\ dx \leq A^{1/p'}B^{1/p}||f||_{L^p}||g||_{L^{p'}}$, as desired.

Now, let's go back and remove the simplifying assumption. As stated previously, we need to change nothing to replace $\mathbb{R}$ with X and Y and dx, dy with $d\mu(x), d\nu(y)$. We do need to make some changes to account for u and w. But we start the same is the simplified case, applying duality to conclude that we need to prove $\int_{X\times Y} |K(x,y)||f(y)||g(x)|\ d\nu(y)\ d\mu(x) \leq A^{1/p'}B^{1/p}||f||_{L^p(d\nu)}||g||_{L^{p'}(d\mu)}$. Now, we need the u and w and appear somehow, so we will simply multiplying inside the integral by $1 = \frac{u^{1/p}(x)}{w^{1/p}(y)}\frac{w^{1/p}(y)}{u^{1/p}(x)}$. We now want to bound $\int_{X\times Y} |K(x,y)||f(y)|\frac{u^{1/p}(x)}{w^{1/p}(y)}|g(x)|\frac{w^{1/p}(y)}{u^{1/p}(x)}\ d\nu(y)\ d\mu(x)$. Once again, we will use Hölder's inequality to see that

$$\int_{X\times Y} |K(x,y)||f(y)|\frac{u^{1/p}(x)}{w^{1/p}(y)}|g(x)|\frac{w^{1/p}(y)}{u^{1/p}(x)}\ d\nu(y)\ d\mu(x) \leq$$

$$\left(\int_{X\times Y} |K(x,y)||f(y)|^p\frac{u(x)}{w(y)}\ d\mu(x)\ d\nu(y)\right)^{1/p}\left(\int_{X\times Y} |K(x,y)||g(x)|^{p'}\frac{w^{p'/p}(y)}{u^{p'/p}(x)}\ d\mu(x)\ d\nu(y)\right)^{1/p'}$$

For the first integral we will, as previously, integrate in x first. The given bound tells us that

$$\left(\int_{X\times Y} |K(x,y)||f(y)|^p\frac{u(x)}{w(y)}\ d\mu(x)\ d\nu(y)\right)^{1/p} \leq \left(\int_Y \frac{B|f(y)|^p w(y)}{w(y)}\ dy\right)^{1/p} = B^{1/p}||f||_{L^p}.$$

Similarly (using the fact that $p'/p = \frac{1}{p-1}$), we see the second integral is bounded by $A^{1/p'}||g||_{L^{p'}}$. Putting this all together, we see that $\int_{X\times Y} |K(x,y)||f(y)||g(x)|\ d\nu(y)\ d\mu(x) \leq A^{1/p'}B^{1/p}||f||_{L^p(d\nu)}||g||_{L^{p'}(d\mu)}$, as desired.

Exercise 33. Let f_n be a sequence of continuous functions on $I = [0,1]$. Suppose that for every $x \in I$ there exists an $M(x) < \infty$ so that $|f_n(x)| \leq M(x)$ for all $n \in \mathbb{N}$. Show then that $\{f_n\}$ is uniformly bounded on some interval, that is there exists $M \in \mathbb{R}$ and an interval $(a,b) \subset I$ so that $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and $x \in (a,b)$.

Solution 33. This is a Baire category theorem problem. I don't have any helpful advice on seeing that that is the right tool to use, but hopefully it will come with practice.

On to the solution. Suppose the desired conclusion does not hold, we will attempt to prove that there is a point x where $|f_n(x)|$ is unbounded. If we want to use Baire's theorem to prove something exists, we should prove that it is contained in a countable intersection of open dense sets. Let $A_M = \{x \in I : |f_n(x)| > M \text{ for some } n\}$. Since each f_n is continuous, A_M the union of the open sets $f_n^{-1}((-\infty,-M)\cup(M,\infty))$ over $n \in \mathbb{N}$ and hence is itself open. Moreover, each A_M is dense, since we have assumed that any interval contains a point x where $|f_n(x)| > M$ for some n. Then $A := \bigcap_{M\in\mathbb{N}} A_M$ is non-empty, by Baire's theorem. But for any $x \in A$, for any $M \in \mathbb{N}$, $x \in A_M$, so there exists $n \in \mathbb{N}$ such that $|f_n(x)| \geq M$.

Then $M(x) \geq M$ for all M, contradicting our assumption that $M(x) < \infty$. Hence, the desired conclusion must hold..

Exercise 34. Suppose that $f_n : [0,1] \to \mathbb{R}$ is a sequence of continuous functions each of which has continuous first and second derivatives on $(0,1)$. Prove: If

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \text{for all} \quad x \in [0,1]$$

and

$$\sup_{n\geq 1} \max_{0<x<1} |f_n''(x)| < \infty,$$

then f' exists and is continuous on $(0,1)$.

Solution 34. We will use Arzela-Ascoli on f_n' to prove that it has a uniformly convergent subsequence, then prove that if $f_n \to f$ uniformly and $f_n' \to g$ uniformly, then f is differentiable and $f' = g$.

Let $\sup_{n\geq 1} \max_{0<x<1} |f_n''(x)| = M$. Equicontinuity is easy: for any $x, y \in [0,1]$, $|f_n'(x) - f_n'(y)| \leq |x-y| \max_{x<\xi<y} |f_n''(\xi)| \leq M|x-y|$, so f' is equicontinuous.

Uniform boundedness is a little tricker. If $f_n'(0)$ exists and is uniformly bounded, then $|f_n'(x) - f_n'(0)| \leq Mx \leq M$ for all $x \in [0,1]$, by the mean value theorem, so as long as we can make sense of $f_n'(0)$ and prove it is bounded, we are good to go. We know for each n that $f_n'(x)$ forms a Cauchy sequence in x as $x \to 0$, since $|f_n'(x) - f_n'(y)| \leq M|x-y|$, so it must converge as $x \to 0$. Then by the mean value theorem, for any $x \in [0,1]$, there exists $y \in (0,x)$ such that $\frac{f_n(x)-f_n(0)}{x} - f_n'(0) = f_n'(y) - f_n'(0)$, so since $f_n'(y) \to f_n'(0)$ as $x \to 0$, we have that $\lim_{x\to 0} \frac{f_n(x)-f_n(0)}{x} = f_n'(0)$, as desired. Now to prove that $f_n'(0)$ is uniformly bounded, note we can Taylor expand f_n at 0 to conclude that $f_n(1/2) = f_n(0) + \frac{f_n'(0)}{2} + c_n$, where c_n is bounded above by a constant times M (from the remainder form of the Taylor expansion, or the mean value theorem again). Then $f_n'(0) = 2(f_n(1/2) - f_n(0) - c_n)$. Since $f_n(0)$ and $f_n(1/2)$ are convergent sequences, they are bounded, $f_n'(0)$ is a bounded sequence, and hence $f_n'(x)$ is uniformly bounded.

Now we know that $f_{n_k} \to f$ uniformly and $f_{n_k}' \to g$ uniformly, so let's prove the claim I made in the first sentence, that f is differentiable and $f' = g$ (you could probably get away with stating this as a fact, but if you have time and can come up with the proof, it's worth including). I think the easiest way to do this is by proving that the integral of g coincides with $f(x) - f(0)$. Let $G(x) = \int_0^x g(t)\,dt$. Then since uniform limits on compact sets commute with integrals, $G(x) = \lim_{k\to\infty} \int_0^x f_{n_k}'(t)\,dt = \lim_{k\to\infty} f_{n_k}(x) - f_{n_k}(0) = f(x) - f(0)$. Then $f(x) = f(0) + \int_0^x g(t)\,dt$, so by the fundamental theorem of calculus, f is differentiable with derivative g.

Exercise 35. A function $f : U \to \mathbb{R}$ defined on a subset $U \subset \mathbb{R}^n$ is

- *locally bounded* if for all $x \in U$ there exists $\varepsilon, R > 0$ such that $|f(y)| \leq R$ for all $y \in U$ with $|x-y| < \varepsilon$,
- *globally bounded* if there exists $R > 0$ such that $|f(y)| \leq R$ for all $y \in U$.

Prove: If $U \subset \mathbb{R}^n$, then the following are equivalent:

(1) U is compact,
(2) every locally bounded function $f : U \to \mathbb{R}$ is globally bounded.

Solution 35. Let's first prove (1) implies (2). Suppose U is compact and let f be a locally bounded function. For every $x \in U$, we have a ball $B(x, \varepsilon)$ on which f is bounded by R. Let $\mathcal{U}$ be the collection of such balls. This forms an open cover of U, so since U is compact, it has a finite subcover $\mathcal{U}' = \{B(x_1, \varepsilon_1), \dots, B(x_m, \varepsilon_m)\}$, where f is bounded by R_i on $B(x_i, \varepsilon_i)$. Then f is bounded on all of U by $\max\{R_1, \dots, R_m\}$, so f is globally bounded.

Now, let's prove (2) implies (1). We will prove U is closed and bounded. First, let's prove it is closed. Take y in the closure of U and suppose $y \notin U$. Then $f(x) = \frac{1}{|x-y|}$ is locally bounded: for any $x \in U$, let $r = |x - y|$. Then if $|x - z| < r/2$, $|x - y| \leq |x - z| + |z - y|$, so $r/2 \leq |x - y| - |x - z| \leq |z - y|$. Hence, $\frac{2}{r} \geq \frac{1}{|z-y|} = f(z)$, so f is locally bounded. On the other hand, there exists a sequence $x_n \in U$ converging to y. Then $f(x_n) = \frac{1}{|x_n-y|}$ is unbounded, so f is not globally bounded, a contradiction. It follows that U must be closed.

To see that U is bounded, let $f(x) : U \to \mathbb{R}$ be given by $f(x) = |x|$. This is locally bounded, since if $|x - y| \leq 1$, then $|f(y)| \leq |x - y| + |x| \leq |x| + 1$. Then f must be globall bounded by some R, so $U \subset B(0, R)$. Hence, U is bounded and closed, so it must be compact.

Exercise 36. Let $\mathcal{K}$ denote the collection of compact subsets of $[0, 1]$. Define the Hausdorff metric on $\mathcal{K}$ by

$$d(K_1, K_2) = \sup_{x \in K_1} \inf_{y \in K_2} |x - y| + \sup_{x \in K_2} \inf_{y \in K_1} |x - y|.$$

Prove that $(\mathcal{K}, d)$ is a complete metric space.

Solution 36. First, let's check that d is a metric. It is hopefully clear that $d(K, K) = 0$ for any $K \in \mathcal{K}$. If K_1, K_2 are distinct compact sets, then, without loss of generality, there exists $x \in K_1 \setminus K_2$, in which case $d(K_1, K_2) > \inf_{y \in K_2} |y - x| > 0$, so $d(K_1, K_2) = 0$ if and only if $K_1 = K_2$. Since the definition of d is symmetric in it's inputs, $d(K_1, K_2) = d(K_2, K_1)$. Now take $K_1, K_2, K_3 \in \mathcal{K}$. We can bound by the triangle inequality

$$\begin{aligned}
\sup_{x \in K_1} \inf_{y \in K_2} |x - y| &\leq \sup_{x \in K_1} \inf_{y \in K_3} \inf_{z \in K_3} |x - z| + |z - y| \\
&\leq \sup_{x \in K_1} \inf_{z \in K_3} |x - z| + \inf_{y \in K_3} \inf_{z \in K_2} |z - y| \\
&\leq \sup_{x \in K_1} \inf_{z \in K_3} |x - z| + \sup_{z \in K_3} \inf_{y \in K_2} |z - y|.
\end{aligned}$$

By the same reasoning, $\sup_{y \in K_2} \inf_{x \in K_1} |x-y| \leq \sup_{y \in K_2} \inf_{z \in K_3} |z-y| + \sup_{z \in K_3} \inf_{x \in K_1} |x-z|$. Summing these two gives that $d(K_1, K_2) \leq d(K_1, K_3) + d(K_2, K_3)$. Thus, the triangle inequality holds.

Proving that the space is complete is quite a bit trickier. If you know what the lim sup and lim inf of a collection of sets are, you should hope that those will coincide with the limit in the Hausdorff topology, so this gives you an outline for how to approach this problem: prove that if K_n is a Cauchy sequence in $\mathcal{K}$, then it converges to $\limsup K_n$ (I'm guessing $\liminf K_n$ would work as well, but I'll leave that as an exercise for the reader). We actually want to be a little more careful than just taking $\limsup K_n$, we will take our putative limit to be $K = \bigcap_{n \in \mathbb{N}} \overline{\bigcup_{j \geq n} K_j}$. Taking the closure is necessary to ensure K is compact. For example, if $K_n = [0, 1 - 1/n]$, then $\limsup K_n = [0, 1)$, while the actual limit should be $[0, 1]$.

Now, let's prove $K_n \to K$ in the Hausdorff metric. Fix $\varepsilon > 0$ and choose N sufficiently large so that for any $n, m \geq N$, $d(K_n, K_m) < \varepsilon/100$. Take arbitrary $x \in K$, we will prove that $\inf_{y \in K_n} |x - y| < \varepsilon/2$ for $n \geq N$. Since $x \in \overline{\bigcup_{j \geq N} K_j}$, we can find some $m \geq N$ such that for some $y \in K_m$, $|x-y| < \varepsilon/100$. For any $n \geq N$, since $d(K_n, K_m) < \varepsilon/100$, we know $\inf_{z \in K_n} |y-z| < \varepsilon/100$, so there exists $z \in K_n$ such that $|y-z| < \varepsilon/100$. It follows that $|x-z| < |x-y| + |y-z| < \varepsilon/50$. Therefore, $\inf_{x \in K_n} |x-z| < \varepsilon/50$ for any $z \in K$, and hence $\sup_{z \in K} \inf_{x \in K_n} |x-z| < \varepsilon/50$.

Now take $n \geq N$ and $x \in K_n$. Since $d(K_n, K_m) < \varepsilon/100$ for any $m \geq N$, we know that for any $m \geq N$, there exists $y_m \in \overline{B}(x_n, \varepsilon/100)$. The sequence y_m is contained in a compact set, so it has a subsequential limit $y_{m_j} \to z \in \overline{B}(x_n, \varepsilon/100)$. Let's prove that $z \in K$. We need to prove that for any $k \in \mathbb{N}$ and any $\delta > 0$, there exists $a \in \bigcup_{m \geq k} K_m$ such that $|a-z| < \delta$. But we know that for j large enough, $y_{m_j} \in \bigcup_{m \geq k} K_m$ and $|y_{m_j} - z| < \delta$. Hence, $z \in \bigcap_{n \in \mathbb{N}} \overline{\bigcup_{j \geq n} K_j} = K$. Since $z \in \overline{B}(x_n, \varepsilon/100)$, $|z-x| \leq \varepsilon/100$. Therefore, $\inf_{z \in K} |x-z| \leq \varepsilon/100$. Since $x \in K_n$ was arbitrary, we see that $\sup_{x \in K_n} \inf_{z \in K} |x-z| \leq \varepsilon/100$. Finally, we conclude that $d(K_n, K) = \sup_{z \in K} \inf_{x \in K_n} |x-z| + \sup_{x \in K_n} \inf_{z \in K} |x-z| < \varepsilon$, as desired.

Exercise 37. Prove that any open set $U \subset \mathbb{R}^n$ can be expressed as a countable union of rectangles.

Solution 37. Let $\mathcal{U} = \{\prod_{i=1}^n (a_i, b_i) \subset U : a_i, b_i \in \mathbb{Q}\}$. Since $\mathbb{Q}$ is countable, $\mathcal{U}$ is countable. For any point $x \in U$, there exists $r > 0$ such that $B(x, r) \subset U$. We can find $q \in \mathbb{Q}$ with $|q - x| < r/100$ and $r_0 \in \mathbb{Q}$ with $r_0 \in [0, r/10]$. Then $R = \prod_{i=1}^n (q_i - r_0, q_i + r_0) \in \mathcal{U}$, $q \in R$, and $R \subset B(x, r) \subset U$. Then $U = \bigcup_{R \in \mathcal{U}} R$ and $\mathcal{U}$ is countable, as desired.

Exercise 38. Suppose that $f \in L^1(\mathbb{R})$. Consider the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = \int_{\mathbb{R}} \frac{f(y)}{1 + |xy|} \, dy.$$

(1) Prove that F is continuous,
(2) Prove that if $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, then $F \in L^p(\mathbb{R})$ for all $p > 2$.

Solution 38.

(1) We will use dominated convergence to prove that F is continuous. We want to prove $\lim_{x \to a} F(x) = F(a)$. Since $\lim_{x \to a} \frac{f(y)}{1+|xy|} = \frac{f(y)}{1+|ay|}$ and $\frac{|f(y)|}{1+|xy|}$ is uniformly bounded by the integrable function $|f(y)|$, the conditions for dominated convergence are satisfied and hence $\lim_{x \to a} F(x) = F(a)$. Therefore, F is continuous.

(2) Since we have $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, let's use Hölder's inequality on $\int_{\mathbb{R}} \frac{f(y)}{1+|xy|} \, dy$ in two different ways and see what comes out. First, let's put an L^1 norm on f and an L^∞ norm on $\frac{1}{1+|xy|}$. This gives that $|F(x)| \leq \int_{\mathbb{R}} \frac{|f(y)|}{1+|xy|} \, dy \leq ||f||_{L^1(\mathbb{R})}$, since $\sup_y \frac{1}{1+|xy|} = 1$. This bound never blows up, but it won't be helpful over all of $\mathbb{R}$, because it is constant. Now, let's put an L^2 norm on f and on $\frac{1}{1+|xy|}$. We need to calculate $\int_{\mathbb{R}} \frac{1}{(1+|xy|)^2} \, dy$, which might be a little tricky to do exactly, but if we do the substitution $u = |x|y$, then we realize that (so long as $x \neq 0$), $\int_{\mathbb{R}} \frac{1}{(1+|xy|)^2} \, dy = \frac{C}{|x|}$, where $C = \int_{\mathbb{R}} \frac{1}{(1+u)^2} \, du$. Then putting this for $||\frac{1}{1+|xy|}||_{L_y^2}^2$, we see that $|F(x)| \leq \frac{C||f||_{L^2}}{\sqrt{|x|}}$. This bound has decay in x, but blows up when x goes to 0.

So we have two bounds for F, one which constant in x and one which has decay in x but a singularity at 0. We can combine these to prove $\int_{\mathbb{R}} |F(x)|^p \, dx < \infty$ for $p > 2$

by using the constant bound in a neighborhood $[-1,1]$ of 0 and the decaying bound everywhere else. Specifically, since $p > 2$, $\int_{|x|>1} \frac{1}{|x|^{p/2}}\, dx < \infty$, so

$$\int_{\mathbb{R}} |F(x)|^p\, dx \leq \int_{-1}^{1} |F(x)|^p\, dx + \int_{|x|>1} |F(x)|^p\, dx \leq 2||f||_{L^1}^p + C||f||_{L^2}^p \int_{|x|>1} \frac{1}{|x|^{p/2}}\, dx < \infty.$$

Therefore, $F \in L^p$ for all $p > 2$.

Exercise 39. For any $n \geq 1$, show that there exists closed sets $A, B \subset \mathbb{R}^n$ with $|A| = |B| = 0$, but $|A + B| > 0$ (as usual $A + B = \{a + b : a \in A, b \in B\}$).

Solution 39. First, let's do this in $\mathbb{R}$. Let C denote the standard Cantor set. Recall that we can characterize elements of C by real numbers $x \in [0,1]$ with a ternary representation containing only 0s and 2s (ternary representations are not unique but we only require one representation to be of the desired form, for example $1 = 0.222\cdots \in C$). Then we can characterize elements of $C/2 = \{x \in [0,1] : 2x \in C\}$ as real numbers with ternary representation containing only 0s and 1s. It is well known that the Cantor set has measure 0, so $C/2$ has measure 0 as well. Let $A = B = C/2$, let's prove that $A + B \supset [0,1]$. Take $y \in [0,1]$ and let $0.y_1y_2\ldots$ be a ternary expansion of y, where $y_i \in \{0,1,2\}$. We will construct elements $a \in A, b \in B$ such that $a + b = y$. Let $a = 0.a_1a_2\ldots$, where

$$a_i = \begin{cases} 0 & y_i = 0 \\ 1 & y_i \neq 0 \end{cases}$$

and $b = 0.b_1b_2\ldots$ where

$$b_i = \begin{cases} 0 & y_i \neq 2 \\ 1 & y_i = 2 \end{cases}.$$

Then $a_i + b_i = y_i$ and $a_i, b_i \in \{0,1\}$ for all i, so $a \in A$, $b \in B$, and $a + b = y$, as desired. It follows that $A + B = [0,1]$. Since $|[0,1]| = 1$, we know that $|A + B| \geq 1 > 0$.

Now, for $\mathbb{R}^n$. Let $A_1, B_1 \subset [0,1]$ denote the sets constructed in the past paragraph. Let $A = A_1 \times [0,1]^{n-1}, B = B_1 \times [0,1]^{n-1}$. The Cartesian product of a measure zero set with any other set has measure zero, so $|A| = |B| = 0$. Now, let's prove that $A + B \supset [0,1]^n$. Take $y = (y^1, \ldots, y^n) \in [0,1]^n$. As previous proven, we can find $a^1 \in A_1, b^1 \in B_1$ such that $a^1 + b^1 = y^1$. Set $a = (a^1, 0, \ldots, 0) \in A$ and $b = (b^1, y^2, \ldots, y^n) \in B$. Then $a + b = y$, so $y \in A + B$. Hence, $|A + B| \geq 1 > 0$.

Exercise 40. Suppose $E \subset \mathbb{R}^d$ is a given set and O_n is the open set $O_n = \{x : d(x, E) < 1/n\}$.

(1) Show that if E is compact, then $|E| = \lim_{n\to\infty} |O_n|$.
(2) Is the statement false for E closed and unbounded?
(3) Is the statement false for E open and bounded?

Solution 40.

(1) We have that $|E| = \int_{\mathbb{R}^d} \chi_E(x)\, dx$ and $|O_n| = \int_{\mathbb{R}^n} \chi_{O_n}(x)\, dx$. Since E is compact, it is contained in $B_R(0)$ for R sufficiently large. It follows that $O_i \subset B_{R+1}(0)$, so $\int_{\mathbb{R}^d} \chi_{O_n}(x)\, dx < \infty$ for all n. Let's prove that $\lim_{n\to\infty} \chi_{O_n}(x) = \chi_E(x)$ for all x. If $x \in E$, then $\chi_{O_n}(x) = \chi_E(x) = 1$ for all x, and hence $\lim_{n\to\infty} \chi_{O_n}(x) = 1 = \chi_E(x)$. If $x \notin E$, then since E is compact, $d(x, E) = c > 0$. Now take N such that $\frac{1}{N} < c$. Then $x \notin O_n$ for any $n \geq N$, so $\chi_{O_n}(x) = 0$ for all $n \geq N$. It follows that

$\lim_{n\to\infty} \chi_{O_n}(x) = 0 = \chi_E(x)$. Hence, $\lim_{n\to\infty} \chi_{O_n}(x) = \chi_E(x)$ for all x. Then by dominated convergence,

$$\lim_{n\to\infty} |O_n| = \lim_{n\to\infty} \int_{\mathbb{R}^d} \chi_{O_n}(x)\, dx = \int_{\mathbb{R}^d} \lim_{n\to\infty} \chi_{O_n}(x)\, dx = \int_{\mathbb{R}^d} \chi_E(x)\, dx = |E|.$$

(2) The statement is false. Consider $E = \mathbb{Z}$. This is discrete and hence closed, but $|O_n| = \infty$ for all n, while $|E| = 0$, so $\lim_{n\to\infty} |O_n| = \infty \neq |E|$.

(3) The statement is false. First, a counterexample in $\mathbb{R}$. Let C be the 1/4-Cantor set. Convince yourself that C has positive measure (unlike the standard Cantor set) as well as being closed and having has empty interior (like the standard Cantor set). Then $E = [0,1] \setminus C$ is open, dense in $[0,1]$, and has measure < 1. But since E is dense, $O_n = [0,1]$ for all n, and hence $\lim_{n\to\infty} |O_n| = 1 > |E|$ for all n.

A counterexample in higher dimensions is not strictly necessary, but taking the Cartesian product of the $\mathbb{R}$ counterexample with $(0,1)^{n-1}$ will give you one.

Exercise 41. Let $\mathcal{D}'(\mathbb{R})$ denote the space of distributions on $\mathbb{R}$ with the weak-$*$ topology. Determine the limit in $\mathcal{D}'(\mathbb{R})$ of the sequence of functions in $\mathbb{R}$:

$$\lim_{n\to\infty} \sqrt{n} e^{\frac{i}{2} n x^2}.$$

Solution 41. Let $u_n = \sqrt{n} e^{inx^2/2}$. For $f \in C_c^\infty(\mathbb{R})$, we have

$$\begin{aligned}
\langle \hat{u}_n, f\rangle &= \langle u_n, \hat{f}\rangle \\
&= \int \sqrt{n} e^{inx^2/2} \hat{f}\, dx \\
&= \int\int \sqrt{n} e^{inx^2/2} e^{-iux} f(u)\, du\, dx \\
&= \int\int \sqrt{n} e^{in/2(x-u/n)^2} e^{-iu^2/(2n)} f(u)\, dx\, du \\
&= \int \sqrt{n} e^{iny^2/2}\, dy \int e^{-iu^2/(4n)} f(u)\, du \\
&= \int e^{iy^2}\, dy \int e^{-iu^2/(4n)} f(u)\, du.
\end{aligned}$$

Since f is C_c^∞, we can use dominated convergence to conclude the final integral is $\int f(u)\, du = \hat{f}(0)$. We are treating u_n as tempered distribution here, which is justified since each u_n has at most polynomial growth. For now, let us denote $\int e^{iy^2}\, dy = C$. We conclude that $\hat{u}_n \to C\hat{\delta}_0$, so $u_n \to C\delta_0$. To find the exact value of C, you need to be carefuly about how you normalize Fourier transform and then compute $\int e^{iy^2}\, dy = (1+i)\sqrt{\pi/2}$. That computation is likely beyond the scope of the exam (the usual way to do it is contour integration), so I think you are fine to leave it as a constant.

Exercise 42. For $s > \frac{1}{2}$, let $H^s(\mathbb{R}^n)$ denote the Sobolev space

$$H^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2\, d\mu(\xi) < \infty\}$$

(where μ is the Lebesgue measure and $\hat{f}$ is the Fourier transform of f). Show that if $u, v \in H^s(\mathbb{R}^n)$ for $s > n/2$, the $uv \in H^s(\mathbb{R}^n)$ and

$$||uv||_{H^s(\mathbb{R}^n)} \leq C||u||_{H^s(\mathbb{R}^n)}||v||_{H^s(\mathbb{R}^n)}$$

for a constant C depending only on s and n.

Solution 42. Note that $\hat{uv} = \hat{u} * \hat{v}$. By Young's inequality, $|\hat{u} * \hat{v}|(\xi) \leq ||\hat{u}||_{L^2}||\hat{v}||_{L^2}$. Then if $C = \int_{\mathbb{R}^n}(1 + |\xi|^2)^s\, d\xi$ (which converges because $s > n/2$),

$$||uv||_{H^s}^2 = \int_{\mathbb{R}^n}(1+|\xi|^2)^s|\widehat{uv}|^2(\xi)\, d\xi = \int_{\mathbb{R}^n}(1+|\xi|^2)^s|\hat{u}*\hat{v}|^2(\xi)\, d\xi \leq C||\hat{u}||_{L^2}^2||\hat{v}||_{L^2}^2 \leq C||u||_{H^s}^2||v||_{H^s}^2.$$

The final inequality uses the fact that $||f||_{L^2} \leq ||f||_{H^s}$ for all $f \in H^s$, which follows from the fact that $(1 + |\xi|^2)\hat{f}(\xi) \geq \hat{f}(\xi)$ for all ξ. Hence, $||uv||_{H^s(\mathbb{R}^n)} \leq C||u||_{H^s(\mathbb{R}^n)}||v||_{H^s(\mathbb{R}^n)}$.

Exercise 43. Let $x_1, \ldots, x_{n+1}$ be pairwise distinct real numbers. Prove that there exists $C > 0$ such that: if $P : \mathbb{R} \to \mathbb{R}$ is a polynomial with degree at most n, then

$$\max_{x\in[0,1]} |P(x)| \leq C \max\{|P(x_1)|, \ldots, |P(x_{n+1})|\}.$$

Solution 43. This follows almost immediately from the fact that all norms on a finite dimensional vector space are equivalent, but I think that might be too powerful of a result for you to be allowed to use. I will give a direct proof instead. The open mapping theorem would also let me skip some steps, but is certainly not necessary for this proof.

Let $\mathcal{P}_n = \{P : \mathbb{R} \to \mathbb{R} \text{ a polynomial} : \deg(P) \leq n\}$ and equip this space with the sup-norm $|| \cdot ||_{L^\infty}$. Define the linear map $L : \mathcal{P}_n \to \mathbb{R}^{n+1}$ by $L(P) = (P(x_1), \ldots, P(x_{n+1}))$. This is an injective function, since a degree n polynomial that vanishes at $n + 1$ points must be 0. Since $\dim(\mathbb{R}^{n+1}) = \dim(\mathcal{P}_n) = n + 1$, L is a bijection. Therefore, it has an inverse $L^{-1} : \mathbb{R}^{n+1} \to \mathcal{P}_n$ (it's not too hard to construct this explicitly, but probably longer to do so than to use argument I gave for it's existence). Linear maps are necessarily continuous, so $||P||_{L^\infty} = ||L^{-1}(L(P))||_{L^\infty} \leq C|L(P)|$, where $| \cdot |$ denotes the standard norm on $\mathbb{R}^{n+1}$. But $|L(P)| = \sqrt{P(x_1)^2 + \cdots + P(x_{n+1})^2} \leq (n + 1)\max\{|P(x_1)|, \ldots, |P(x_{n+1})|\}$. Hence, $||P||_{L^\infty} \leq (n + 1)C \max\{|P(x_1)|, \ldots, |P(x_{n+1})|\}$, as desired.

Exercise 44. Given a real number x, let $\{x\}$ denote the fractional part of x. Suppose α is an irrational number and define $T : [0, 1] \to [0, 1]$ by

$$T(x) = \{x + \alpha\}.$$

Prove: If $A \subset [0, 1]$ is measurable and $T(A) = A$, then $|A| \in \{0, 1\}$.

Solution 44. If $|A| = 0$ or $|A| = 1$, we are done. So assume $|A| = c \in (0, 1)$. Then A has a Lebesgue point for χ_A, that is, a point $x \in (0, 1)$ where $\lim_{r\to 0} \frac{m([x-r,x+r]\cap A)}{2r} = 1$. Fix r_0 and sufficiently small so that $m([x-r_0, x+r_0]\cap A) > 2r_0\varepsilon$, where ε is a number to be determined later. We will use this to prove that $|A| > c$, a contradiction.

Next we will prove that T is measure preserving on all measurable sets. This is easy to check for intervals. Since $T([0, 1]) = [0, 1]$ and T is a bijection, if T is measure preserving on a set, it is also measure preserving on it's complement. Finally, if T is measure preserving on a collection of sets, it is measure preserving on their disjoint union. It follows by the $\pi - \lambda$ theorem that T is measure preserving on all measurable sets.

Since T is injective, $T(A \cap B) = T(A) \cap T(B)$ for any sets A, B. And T is measure preserving, so $m(T(A)) = m(A)$ for any set A. Then for any interval I, $m(I \cap A) = m(T(I \cap A)) = m(T(I) \cap T(A)) = m(T(I) \cap A)$. We can iterate this to prove that $m(I \cap A) = m(T^n(I) \cap A)$. Now if $I = [x - r_0, x + r_0]$ and $|T^n(x) - x| > 2r_0$, then $T^n([x - r_0, x + r_0]) \cap [x - r_0, x + r_0] = \emptyset$, as both are intervals with length r_0 and center seperated by r_0. Assume we can find $k = \lfloor \frac{1}{2r_0} \rfloor - 1$ values $0 = n_1, \ldots, n_k$ such that $|T^{n_j}(x) - T^{n_i}(x)| > 2r_0$ for all $j \neq i$. It follows that $m(A) \geq \sum_{i=1}^{m} m(T^{n_i}(I) \cap A) \geq m2r_0\varepsilon \geq 2r_0(\frac{1}{2r_0} - 2)\varepsilon \geq (1 - 4r_0)\varepsilon$. By choosing ε to be very close to 1 and then r_0 very close to 0, we can ensure $m(A) > c$, a contradiction.

We will conclude by proving our assumption that we can find k well-spread out points. It suffices to prove that $\{T^n(x) : n \in \mathbb{N}\}$ is dense for any x. The mapping $q : \mathbb{R} \to \mathbf{T}$ sending x to $e^{2\pi i x}$ is an open map, so the preimage of a dense set is dense, and since q restricts to a bijection on $[0, 1)$, $\{T^n(x) : n \in \mathbb{N}\}$ is dense in $[0, 1]$ as long as $q(\{T^n(x) : n \in \mathbb{N}\})$ is dense in $\mathbf{T}$. But $q(\{x\}) = q(x)$, so $q(T^n(x)) = q(x + n\alpha) = e^{2\pi i(x+n\alpha)}$. Therefore, $q(\{T^n(x) : n \in \mathbb{N}\}) = \{e^{2\pi i(x+\alpha n)} : n \in \mathbb{Z}\}$. Since multiplication by $e^{2\pi i x}$ is an automorphism of $\mathbf{T}$, we may assume $x = 0$, so we need only prove that $G = \{e^{2\pi i \alpha n} : n \in \mathbb{Z}\}$ is dense in $\mathbf{T}$. But G is an infinite subgroup of $\mathbf{T}$, so following the proof in the solution to exercise 2, we see that it is dense.

Jacob Denson has a more direct solution to this problem in his notes.

Exercise 45. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable, real-valued functions on a measure space X such that $f_n \to f$ pointwise as $n \to \infty$, where $f : X \to \mathbb{R}$, and suppose that for some constant $M > 0$,

$$\int |f_n| \, d\mu \leq M \qquad \text{for all } n \in \mathbb{N}.$$

(1) Prove that

$$\int |f| \, d\mu \leq M.$$

(2) Give an example to show that we may have $\int |f_n| \, d\mu = M$ for every $n \in \mathbb{N}$, but $\int |f| \, d\mu < M$.

(3) Prove that

$$\lim_{n\to\infty} \int ||f_n| - |f| - |f_n - f|| \, d\mu = 0.$$

Solution 45.

(1) By Fatou's lemma, $\int |f| \, d\mu \leq \liminf \int |f_n| \, d\mu \leq M$.

(2) Take $f_n(x) = \chi_{[n,n+M]}$. Then $f_n \to 0$ pointwise everywhere and $\int |f_n|(x) \, dx = M$ for all n, but $\int 0 \, dx = 0 < M$.

(3) We will use dominated convergence. Since $f_n \to f$ pointwise, $|f_n| - |f| - |f_n - f| \to 0$ pointwise, so as long as the conditions for dominated convergence are satisfied, we will have $\lim_{n\to\infty} \int ||f_n| - |f| - |f_n - f|| \, d\mu = 0$. We just need to find an integrable for $||f_n| - |f| - |f_n - f||$. We want that upper bound to be an integrable function, and given part (1), a reasonable guess is $|f|$ or some multiple of $|f|$. By rearranging the triangle inequality, we know $|f_n - f| \geq |f_n| - |f|$, so $||f_n| - |f| - |f_n - f|| = |f_n - f| - |f_n| + |f|$. By the same reasoning, $|f_n - f| - |f_n| \leq |f|$, so $|f_n - f| - |f_n| + |f| \leq 2|f|$. Hence $||f_n| - |f| - |f_n - f|| \leq 2|f|$, and by part (1), we know $\int 2|f| \, d\mu \leq 2M$, so the

conditions for the dominated convergence theorem hold. As discussed previously, this completes the problem.

Exercise 46. Consider the following equation for an unknown function $f : [0,1] \to \mathbb{R}$:

$$f(x) = g(x) + \lambda \int_0^1 (x-y)^2 f(y)\ dy + \frac{1}{2}\sin(f(x)).$$

Prove that there exists a number λ_0 such that for all $\lambda \in [0, \lambda_0)$ and all continuous functions g on $[0,1]$, the equation has a continuous solution.

Solution 46. This is, of course, a contraction mapping problem. Let $T(f)(x) = g(x) + \lambda \int_0^1 (x-y)^2 f(y)\ dy + \frac{1}{2}\sin(f(x))$. Then

$$\begin{aligned} d(T(f_1), T(f_2)) &\leq \sup_{x\in[0,1]} \lambda \int_0^1 (x-y)^2 |f_1(y) - f_2(y)|\ dy + \frac{1}{2}|\sin(f_1(x)) - \sin(f_2(x))| \\ &\leq \sup_{x\in[0,1]} \lambda \int_0^1 (x-y)^2 |f_1(y) - f_2(y)|\ dy + \sup_{x\in[0,1]} \frac{1}{2}|\sin(f_1(x)) - \sin(f_2(x))|. \end{aligned}$$

Since $|\sin'(x)| \leq 1$ for all x, $\frac{1}{2}|\sin(f_1(x)) - \sin(f_2(x))| \leq \frac{1}{2}|f_1(x) - f_2(x)|$, so

$$\sup_{x\in[0,1]} \frac{1}{2}|\sin(f_1(x)) - \sin(f_2(x))| \leq \frac{1}{2} d(f_1, f_2).$$

Then it suffices to choose to chose λ small enough so that $\lambda \int_0^1 (x-y)^2 |f_1(y) - f_2(y)|\ dy \leq \frac{1}{3} d(f_1, f_2)$. To see that this can be done, note that $\int_0^1 (x-y)^2 |f_1(y) - f_2(y)|\ dy \leq d(f_1, f_2) \int_0^1 (x-y)^2\ dy = d(f_1, f_2)\frac{x^3-(x-1)^3}{3}$. By the mean-value theorem, $\frac{x^3-(x-1)^3}{3} \leq \sup_{x\in[-1,1]} x^2 \leq 1$. Therefore, $\int_0^1 (x-y)^2 |f_1(y) - f_2(y)|\ dy \leq d(f_1, f_2)$, so if we take $\lambda_0 = \frac{1}{3}$, then if $\lambda \leq \lambda_0$, then $\lambda \int_0^1 (x-y)^2 |f_1(y) - f_2(y)|\ dy \leq \frac{1}{3} d(f_1, f_2)$ and hence $d(T(f_1), T(f_2)) \leq \frac{5}{6} d(f_1, f_2)$. Then T is a contraction mapping, so it has a fixed point, which necessarily solves the given equation.

It is part of the contraction mapping theorem that such a fixed point is unique, but it is quite easy to prove as well. Suppose T has two fixed points f_1, f_2. Then since T is a contraction, $d(T(f_1), T(f_2)) < (1-\varepsilon) d(f_1, f_2)$ for some $\varepsilon \in (0,1)$, but $d(T(f_1), T(f_2)) = d(f_1, f_2)$, a contradiction unless both are zero. Hence, the fixed point is unique.

Exercise 47. Given $\alpha \geq 0$ the α-dimensional Hausdorff measure of a set $X \subset \mathbb{R}^n$ is

$$\mathcal{H}^\alpha(X) = \lim_{r\to 0} \inf\{\sum_{i=1}^\infty r_i^\alpha : X \subset \bigcup_{i=1}^\infty B(x_i, r_i), r_i < r \text{ for all } i\}$$

and the Hausdorff dimension is $\dim_H(X) = \inf\{\alpha \geq 0 : \mathcal{H}^\alpha(X) = 0\}$.

Prove the following:

(1) If $X \subset \mathbb{R}^n$ and μ is a finite Borel measure on $\mathbb{R}^n$ such that $\mu(X) > 0$ and $\mu(B(x,r)) \leq r^\alpha$ for all open balls $B(x,r)$, then $\dim_H(X) \geq \alpha$.
(2) If $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is the unit circle, the $\dim_H(S^1) = 1$.

Solution 47.

(1) We need to show that $\mathcal{H}^\alpha(X) > 0$. Suppose otherwise. Then for any $\varepsilon > 0$, there exists a collection of balls $B(x_1, r_1), B(x_2, r_2), \dots$ covering X with $\sum_{i=1}^\infty r_i^\alpha < \varepsilon$. Then

$$\mu(X) \le \mu\left(\bigcup_{i=1}^\infty B(x_i, r_i)\right) \le \sum_{i=1}^\infty \mu(B(x_i, r_i)) \le \sum_{i=1}^\infty r_i^\alpha < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\mu(X) = 0$, a contradiction. Hence, $\mathcal{H}^\alpha(X) > 0$, so $\dim_H(X) \ge \alpha$.

(2) First, let's check that $\dim_H(S^1) \le 1$. To do so, we need to prove that $\mathcal{H}^s(S^1) = 0$ for any $s > \alpha$. For any $r > 0$, let $x_i = (\cos(\theta_i), \sin(\theta_i))$, where $\theta_i = \frac{2\pi i r}{100}$, for $i = 1, 2, \dots, \lfloor \frac{100}{2\pi r}, \rfloor$. Then for any $x \in S^1$, $x = (\cos(\theta), \sin(\theta))$ for some $\theta \in [0, 2\pi]$, so if θ_i is the closest point to θ, then $|x - x_i| \le |\cos(\theta) - \cos(\theta_i)| + |\sin(\theta) - \sin(\theta_i)| \le 2|\theta - \theta_i| \le \frac{1}{100r}$. Hence, $x \in B(x_i, r)$, so $S^1 \subset \bigcup_{i=1}^{\lfloor \frac{100}{2\pi r} \rfloor} B(x_i, r_i)$. Moreover, $\sum_{i=1}^{\lfloor \frac{100}{2\pi r}} r_i^s \le \frac{100}{2\pi r} r^s = \frac{100}{2\pi} r^{s-1}$. Since $s - 1 > 0$, $\lim_{r \to 0} \frac{100}{2\pi} r^{s-1} = 0$, and hence $\mathcal{H}^s(S^1) = 0$. Since $s > \alpha$ was arbitrary, we see that $\dim_H(S^1) \le 1$.

Now let's prove that $\dim_H(S^1) \ge 1$. We will use the previous part and radial integration to accomplish this. Define $\mu(A) = \frac{1}{100} \int_{S^1} \chi_A(\cos(\theta), \sin(\theta))\, d\theta$. This is a well-defined measure, you can either check this from the definition or recall that μ is the pushforward of the Lebesgue measure on $\mathbb{S}^1$ under the map $\theta \mapsto (\sin(\theta), \cos(\theta))$. Also, $\mu(S^1) = 2\pi > 0$, so it remains to check the measure condition for balls. For a ball $B(x, r)$, if $B(x, r) \cap S^1 = \emptyset$, then $\mu(B(x, r)) = 0 < r$. If $B(x, r) \cap S^1 \ne \emptyset$, then $\mu(B(x, r))$ is $\frac{1}{100}$ times length of the circular arc $B(x, r) \cap S^1$. Let θ_1, θ_2 be the angles at the endpoints of arc and note that the length of the circular arc is $|\theta_1 - \theta_2|$. Then $\mu(B(x, r)) = \frac{|\theta_1 - \theta_2|}{100}$. On the other hand, $(\cos(\theta_1), \sin(\theta_1)), (\cos(\theta_2), \sin(\theta_2)) \in B(x, r)$ and $B(x, r)$ is a convex set, so it contains a line of length $\sqrt{|\cos(\theta_1) - \cos(\theta_2)|^2 + |\sin(\theta_1) - \sin(\theta_2)|^2} \ge \frac{|\theta_1 - \theta_2|}{2}$ (one of the approximations $|\sin(\theta_1) - \sin(\theta_2)| \ge \frac{|\theta_1 - \theta_2|}{2}$ and $|\cos(\theta_1) - \cos(\theta_2)| \ge \frac{|\theta_1 - \theta_2|}{2}$ must hold for any value of $\theta_1, \theta_2 \in [0, 2\pi)$). The ball then must have radius $> \frac{|\theta_1 - \theta_2|}{4}$, so

$$\mu(B(x, r)) = \frac{|\theta_1 - \theta_2|}{100} \le \frac{|\theta_1 - \theta_2|}{4} < r.$$

Therefore, by the first part, $\dim_H(S^1) \ge 1$, and hence $\dim_H(S^1) = 1$.

Exercise 48. Let $X = [0, 1]$ with Lebesgue measure and $Y = [0, 1]$ with counting measure. Give an example of a measurable function $f : X \times Y \to [0, \infty)$ for which Fubini's theorem does not apply. (This example shows that the theorem is not valid if the hypothesis of σ-finiteness is omitted.)

Solution 48. Let $f(x, y) = \begin{cases} 1 & x = y \\ 0 & \text{otherwise} \end{cases}$. Then $\int_X \int_Y f(x, y)\, dy\, dx = \int_X 1\, dx = 1$, while $\int_Y \int_X f(x, y)\, dx\, dy = \int_Y 0\, dy = 0$.

Exercise 49. For $s > \frac{1}{2}$ let $H^s(\mathbb{R}^n)$ denote the Sobolev space

$$H^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2\, d\mu(\xi) < +\infty\}$$

(where μ is the Lebesgue measure and $\hat{f}$ is the Fourier transform of f). Use the Fourier transform to prove that if $u \in H^s(\mathbb{R}^n)$ for $s > n/2$, then $u \in L^\infty(\mathbb{R}^n)$, wih the bound

$$||u||_{L^\infty} \leq C||u||_{H^s(\mathbb{R}^n)}$$

for a constant C depending only on s and n.

Solution 49. Recall that $||u||_{L^\infty} \leq ||\hat{u}||_{L^1}$. Let's write $\hat{u}(\xi) = (1+|\xi|^2)^{s/2}(1+|\xi|^2)^{-s/2}\hat{u}(\xi)$. Then by Hölder's inequality, if we let $C = |(|1+|\xi|)^{-s/2}||_{L^2}$, which is finite because $s > n/2$, we see that

$$||\hat{u}||_{L^1} \leq ||(1+|\xi|^2)^{-s/2}||_{L^2}||(1+|\xi|^2)^{s/2}\hat{u}(\xi)||_{L^2} = C||u||_{H^s(\mathbb{R}^n)}.$$

Hence, $||u||_{L^\infty} \leq C||u||_{H^s(\mathbb{R}^n)}$, as desired.

Exercise 50. Assume that X is a compact metric space and $T : X \to X$ is a continuous map. Let $\mathcal{M}_1(T)$ denote the set of Borel probability measures on X such that $T_*\mu = \mu$. Prove:

(1) $\mathcal{M}_1(T) \neq \emptyset$.
(2) If $\mathcal{M}_1(T) = \{\mu\}$ consists of a single measure μ, then

$$\int_X f\ d\mu = \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f \circ T^n(x)$$

for every continuous function $f : X \to \mathbb{R}$ and point $x \in X$.

Solution 50.

(1) For any $x \in X$, denote by δ_x the unit mass at x, that is, the measure satisfying $\int f(y)\ d\delta_x(y) = f(x)$. Now fix a point $x \in X$, let m_n be the Borel measure $\delta_{T^n(x)}$ for $n = 0, 1, \ldots,$ and for $N = 0, 1, \ldots,$ let $\mu_N = \frac{1}{N}\sum_{n=0}^{N-1} m_n$. Then since each m_n is a probability measure, $||\mu_N|| = \frac{1}{N}\sum_{n=0}^{N-1} ||m_n|| = 1$, so μ_N is also a probability measure. Then by Banach-Alaoglu it has a weak-$*$ convergent subsequence. Call the limit μ. Since X is compact, $1 \in C_0(X)$, so $\int 1\ d\mu = \lim_{k\to\infty}\int 1\ d\mu_{N_k} = 1$. We also know that μ is a positive measure, since if $f \in C(X)$ is non-negative, then $\int f\ d\mu = \lim_{k\to\infty}\int f\ d\mu_{N_k} \geq 0$, so since μ is a positive measure with unit mass, it is a probability measure. Finally, for any $f \in C(X)$, we have

$$\begin{aligned}\left|\int f\circ T\ d\mu - \int f\ d\mu\right| &= \lim_{k\to\infty}\left|\int f\circ T - f\ d\mu_{N_k}\right| \\ &= \lim_{k\to\infty}\frac{1}{N_k}\left|\sum_{n=0}^{N_k-1} f\circ T^{n+1}(x) - \sum_{n=0}^{N_k-1} f\circ T^n(x)\right| \\ &= \lim_{k\to\infty}\frac{|f(T^{n+1}(x)) - f(x)|}{N_k} \\ &= 0.\end{aligned}$$

Therefore $T_*\mu = \mu$, so $\mu \in \mathcal{M}_1(T)$.

(2) Denote the measure constructed in the previous part is μ_x, where x is the point we started at. If $\mathcal{M}_1(T)$ consists of a single measure μ, then $\mu = \mu_x$ for all $x \in X$. It therefore suffices to prove that $\int_X f\ d\mu_x = \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} f\circ T^n(x)$ for all $f \in C(X)$. Fix $f \in C(X)$. By construction, $\int_X f\ d\mu_x = \lim_{k\to\infty}\frac{1}{N_k}\sum_{n=0}^{N_k-1} f\ \circ T^n(x)$ for

some sequence $N_k \to \infty$. Now let's prove that $\int f\ d\mu = \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f \circ T^n(x)$. Keeping with the notation of first part, note that for any subsequence $N_j \to \infty$, $\frac{1}{N_j}\sum_{n=0}^{N_j-1} f \circ T^n(x) = \int f\ d\mu_{N_j}$. Following the proof in part 1, we μ_{N_j} has a subsequence $\mu_{N_{j_k}} \to \mu' \in \mathcal{M}_1(T)$, so $\lim_{k\to\infty} \frac{1}{N_{j_k}}\sum_{n=0}^{N_{j_k}-1} f \circ T^n(x) = \int f\ d\mu'$. But since $\mathcal{M}_1(T)$ consists of a single element, $\mu' = \mu$, so $\int f\ d\mu' = \int f\ d\mu$. Therefore, any subsequence of $\frac{1}{N}\sum_{n=0}^{N-1} f \circ T^n(x)$ has a subsubsequence converging to $\int f\ d\mu$, so $\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f \circ T^n(x) = \int f\ d\mu$, as desired.

Exercise 51. Find the Fourier transform of the following function: $f \in \mathbb{R}^2$:

$$f(x) = e^{ix\xi_0}|x - x_0|^{-1}.$$

Solution 51. First, let's suppose $\xi_0 = x_0 = 0$. Then since f is a radial function homogenous of degree -1, $\hat{f}$ is radial and homogenous of degree $(-2-(-1)) = -1$. Since every radial, homogenous function of degree m function is $C|x|^m$ for some constant C, it follows that $\hat{f}(\xi) = \frac{C}{|\xi|}$. We can find the constant by computing

$$\langle \frac{1}{|x|}, \widehat{e^{-|\xi|^2}} \rangle = \langle \widehat{\frac{1}{|x|}}, e^{-|\xi|^2} \rangle = \langle \frac{C}{|\xi|}, e^{-|\xi|^2} \rangle.$$

We can compute that the Fourier transform of $e^{-|\xi|^2}$ is $\sqrt{\pi}e^{-\pi^2x^2}$, so

$$\sqrt{\pi}\int_{\mathbb{R}^2} \frac{e^{-\pi^2x^2}}{|x|}\ dx = C\int_{\mathbb{R}^2} \frac{e^{-|\xi|^2}}{|\xi|}\ d\xi.$$

Changing variables, we see that $\sqrt{\pi}\int_{\mathbb{R}^2} \frac{e^{-\pi^2x^2}}{|x|}\ dx = \sqrt{\pi}\int_{\mathbb{R}^2} \frac{e^{-x^2}}{|x|}\ dx$ (the scale invariance seen here is a property of the scale invariant differential term $\frac{dx}{x} = d\log(x)$). Hence, $C = \sqrt{\pi}$.

Exercise 52. Let $f \in C^1([0,1])$. Show that for every $\varepsilon > 0$ there exists a polynomial p such that

$$||f - p||_\infty + ||f' - p'||_\infty < \varepsilon.$$

Solution 52. First, note that if $|f' - p'|(x) < \varepsilon/2$ for all x and if $f(0) = p(0)$, then $|f(x) - p(x)| \le \int_0^x |f'(t) - p'(t)|\ dt \le \varepsilon/2$. By Stone-Weierstrass, we can find a polynomial q which satisfies $||q - f'||_{L^\infty} < \varepsilon/2$. Then $p(x) = \int_0^x q(t)\ dt + f(0)$ is a polynomial, satisfies $p(0) = f(0)$, and $||p' - f'||_{L^\infty} < \varepsilon/2$ and therefore satisfies $||f - p||_{L^\infty} < \varepsilon/2$, solving the problem.

Exercise 53. Let $a = (a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers. Prove that the set

$$X = \{(x_n)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N}) : x_n \in [0, a_n] \text{ for all } n \in \mathbb{N}\}$$

is compact in the $\ell^1(\mathbb{N})$ norm if and only if $(a_n)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N})$.

Solution 53. Let's start with the easy direction. Suppose X is compact. Define $a^m \in X$ for $m \in \mathbb{N}$ by $a^m_n = \delta_{n\leq m} a_n$. Then a^m must subsequentially converge to $b \in X \subset \ell^1(\mathbb{N})$, but $b_n = a_n$ for all $n \in \mathbb{N}$ (since $a^{m_j}_n = a_n$ if m_j is our subsequence from compactness and j is sufficiently large), so $b = a$, and hence $a \in \ell^1(\mathbb{N})$.

Now, for the hard direction. When I was taking the quals, I didn't know what a totally bounded set was, so I would have proven this by proving any sequence in X has a convergent subsequence. I am not going to present that argument here, but I invite you to give it a try. If you get stuck, you could consult the proof that totally bounded and closed sets are compact.

But now I know what a totally bounded set is, so I will prove it using that. Assume $a \in \ell^1(\mathbb{N})$. First, note that X is closed. If $a^m \in X$ converges to a limit b, then $a^m_n \to b_n$ for each $n \in \mathbb{N}$. Since each $a^m_n \in [0, a_n]$, it follows that $b_n \in [0, a_n]$ and hence $b \in X$. Now, let's prove X is totally bounded. Fix $\varepsilon > 0$. Since $a \in \ell^1$, there exists $N \in \mathbb{N}$ such that $\sum_{n>N} a_n < \frac{\varepsilon}{2}$. For each $n \leq N$, define a finite collection of reals $b^1_n, \dots b^{j_n}_n \in [0, a_n]$ such that the $\frac{\varepsilon}{2N}$ balls around the elements b^i_n cover $[0, a_n]$. Define the finite collection elements

$$B = \{(b^{j_{i_1}}_1, \dots, b^{j_{i_N}}_N, a_{N+1}, a_{N+2}, \dots) : 1 \leq j_{i_n} \leq j_n \text{ for } 1 \leq n \leq N\}.$$

Let U denote the collection of ε-balls centered at points in B. For any $x \in X$, $\sum_{n>N} |x_n - a_n| < \frac{\varepsilon}{2}$. For each $n < N$, we can find an element $b^{i_n}_n$ within $\frac{\varepsilon}{2N}$ of x_n. Then $b = (b^{i_1}_1, \dots, b^{i_N}_N, a_{N+1}, a_{N+2}, \dots) \in B$ and $||x - b||_{\ell^1} < N\frac{\varepsilon}{2N} + \frac{\varepsilon}{2} \leq \varepsilon$. Therefore, U covers X. Since this can be done for any $\varepsilon > 0$, we know that X is totally bounded, and because closed, totally bounded sets are compact, we are done.

Exercise 54. Let $z_1, z_2, \dots, z_n$ be points on the unit circle $\mathbf{T} = \{|z| = 1\}$ in the complex plane. Let $E \subset \mathbf{T}$ satisfy $m(E) > 2\pi(1 - \frac{1}{n})$. Prove that E can be rotated so that all the points z_k fall into the rotated set, i.e., that there exists $\alpha \in \mathbf{T}$ such that $\alpha z_k \in E$ for $k = 1, 2, \dots, n$.

Solution 54. Let $A_k = \{\alpha \in \mathbf{T} : \alpha z_k \in E\}$, we want to prove that $A_1 \cap \dots \cap A_n \neq \emptyset$. Since $\alpha z_k \in E$ if and only if $\alpha \in z_k^{-1}E$, we see that $A_k = z_k^{-1}E$. Rotations are measure preserving, so $|A_k| = |E| > 2\pi(1 - \frac{1}{n})$. Then if $A_1 \cap \dots \cap A_n = \emptyset$, we can take the set-difference of both sides from $\mathbf{T}$ to see that $(\mathbf{T} \setminus A_1) \cup \dots \cup (\mathbf{T} \setminus A_n) = \mathbf{T}$. Each $\mathbf{T} \setminus A_i$ has measure $< \frac{2\pi}{n}$, so $|(\mathbf{T} \setminus A_1) \cup \dots \cup (\mathbf{T} \setminus A_n)| < 2\pi = |\mathbf{T}|$, a contradiction. Therefore, $A_1 \cap \dots \cap A_n \neq \emptyset$, so there exists $\alpha \in \mathbf{T}$ such that $\alpha z_k \in E$ for all k.

Exercise 55. Let $f \in L^1(\mathbb{R})$ satisfy $\int_a^b f(x)\, dx = 0$ for any two rational numbers $a < b$. Does it follow that $f(x) = 0$ for almost every x?

Solution 55. It does follow that $f(x) = 0$ almost everywhere. Suppose otherwise. Without loss of generality, we may assume there exists a set E of measure $\varepsilon > 0$ on which $f > 1$. By the outer regularity of the Lebesgue measure, there exists a sequence of open sets $V_n \supset E$ such that $m(V_n) \to m(E)$. On $\mathbb{R}$, any open set is a countable collection of open intervals. By lengthening the mth interval in the collection by less than $\frac{1}{n2^m}$, we can ensure each of the open intervals has rational endpoints. Call the new open set U_n and note that $m(U_n) \leq m(V_n) + \frac{1}{n}$, and hence $m(U_n) \to m(E)$, and $U_n \supset V_n \supset E$ for all n. We then see that χ_{U_n} converges to χ_E in measure, and hence $\chi_{U_{n_k}}$ converges χ_E almost everywhere for some subsequence n_k. We know that $\int \chi_{U_{n_k}} f(x)\, dx = 0$, so by dominated convergence, $\int \chi_E f(x)\, dx = 0$ as well, contradicting our assumption that $|f| > 1$ on E.

Exercise 56. Let σ be a Borel probability measure on $[0,1]$ satisfying

(1) $\sigma([1/3,2/3]) = 0$;
(2) $\sigma([a,b]) = \sigma([1-b,1-a])$ for any $0 \leq a < b \leq 1$;
(3) $\sigma([3a,3b]) = 2\sigma([a,b])$ for any a,b such that $0 \leq 3a < 3b \leq 1$.

Complete the following with justification:

(1) Find $\sigma([0,1/8])$.
(2) Calculate the second moment of σ, i.e. the integral

$$\int_0^1 x^2 \, d\sigma(x).$$

Solution 56. I think of σ as the "uniform" measure supported on the Cantor set. That is not explicitly used in my solutions here, but might provide some context for how I came to the solution below.

(1) Since σ is a probability measure, it has total mass 1. The third property implies that $\sigma([0,1/3]) = \frac{\sigma([0,1])}{2} = \frac{1}{2}$, $\sigma([0,1/9]) = \frac{\sigma([0,1/3])}{2} = \frac{1}{4}$ and that $\sigma([1/9,2/9]) = \frac{\sigma([1/3,2/3])}{2} = 0$. It follows that $\sigma([0,1/8]) = \sigma([0,1/9]) + \sigma([1/9,1/8]) = \frac{1}{4}$.
(2) The given properties are listed in terms of the measure σ. For this problem, we need to turn the properties into properties of the integral $\int_0^1 f(x)\, d\sigma(x)$. The first property is immediate, but for the other two, you could formally check this by expressing the properties as $\sigma = T_*\sigma$ for appropriate choices of T, then recalling how to integrate pushforward measures. I don't think you would actually need to do this on the qual, so I will leave that as an exercise.

The properties are as follows:
(a) $\int_{1/3}^{2/3} f(x)\, d\sigma(x) = 0$,
(b) $\int_0^1 f(x)\, d\sigma(x) = \int_0^1 f(1-x)\, dx$, and
(c) $\int_0^1 f(x)\, d\sigma(x) = 2\int_0^{1/3} f(3x)\, d\sigma(x)$.

By the first property

$$\int_0^1 x^2 \, d\sigma(x) = \int_0^{1/3} x^2 \, d\sigma(x) + \int_{2/3}^1 x^2 \, d\sigma(x).$$

By the second property,

$$\int_{2/3}^1 x^2 \, d\sigma(x) = \int_0^{1/3} (1-x)^2 \, d\sigma(x) = \int_0^{1/3} 1 \, d\sigma(x) + \int_0^{1/3} x^2 \, d\sigma(x) - 2\int_0^{1/3} x \, d\sigma(x).$$

So $\int_0^1 x^2 \, d\sigma(x) = 2\int_0^{1/3} x^2 \, d\sigma(x) + \sigma([0,1/3]) - 2\int_0^{1/3} x\, d\sigma(x)$. By the third property, we see that $2\int_0^{1/3} x^2\, d\sigma(x) = \frac{2}{9}\int_0^{1/3} (3x)^2 \, d\sigma(x) = \frac{1}{9}\int_0^1 x^2\, d\sigma(x)$. Substituting this in, we see that $\frac{8}{9}\int_0^1 x^2\, d\sigma(x) = \sigma([0,1/3]) - 2\int_0^{1/3} x\, d\sigma(x)$. In this way, we have reduced integrating x^2 to integrating x. We will proceed similarly to reduce integrating x to integrating 1. For brevity, I will not explain each step, but they all follow from applying the properties above.

We know that

$$\begin{aligned} 2\int_0^{1/3} x\ d\sigma(x) &= \frac{2}{3}\int_0^1 (3x)\ d\sigma(x) \\ &= \frac{1}{3}\int_0^1 x\ d\sigma(x) \\ &= \frac{1}{3}\int_0^{1/3} x\ d\sigma(x) + \frac{1}{3}\int_{2/3}^{3} x\ d\sigma(x) \\ &= \frac{1}{3}\int_0^{1/3} x\ d\sigma(x) + \frac{1}{3}\int_0^{1/3} (1-x)\ d\sigma(x) \\ &= \frac{1}{3}\int_0^{1/3} 1\ d\sigma(x) \\ &= \frac{1}{3}\sigma([0,1/3]). \end{aligned}$$

Hence, $\frac{8}{9}\int_0^1 x^2\ d\sigma(x) = \sigma([0,1/3]) - \frac{1}{3}\sigma([0,1/3])$, and therefore $\int_0^1 x^2\ d\sigma(x) = \frac{3}{4}\sigma([0,1/3])$. Now $\sigma([0,1/3]) = \frac{1}{2}\sigma([0,1])$, by the third property, so since σ has mass one, $\sigma([0,1/3]) = \frac{1}{2}$. Therefore, $\int_0^1 x^2\ d\sigma(x) = \frac{9}{8}\cdot\frac{1}{3} = \frac{3}{8}$.

Exercise 57. Does the improper integral

$$\int_2^\infty \frac{x\sin(e^x)}{x+\sin(e^x)}\ dx$$

converge?

Solution 57. This is a tricky problem. I'll present one solution.

First, let's prove that $\sin(e^x)$ is integrable. I later googled it and found out you can prove this by integrating by parts, but the sublevel set approach we discussed in class works as well (and I already typed it up). Let's estimate the integral over individual half-periods of $\sin(e^x)$. Let $a_k = \log(\pi 2k)$ and $b_k = \log(\pi(2k+1))$ (we can safely ignore the contribution from 2 to a_1). Between a_k and b_k, the integrand is positive and between b_k and a_{k+1}, the integrand is negative. We can analyze $\int_{a_k}^{b_k} \sin(e^x)\ dx$ by u-substitution. Let $u = e^x$, so $\frac{du}{u} = dx$. In this variable $\int_{\pi 2k}^{\pi(2k+1)} \frac{\sin(u)}{u}\ du$. Since sin is always positive in this interval, we see that this is between $\frac{1}{\pi 2k}\int_{\pi 2k}^{\pi(2k+1)} \sin(u)\ du = \frac{1}{k\pi}$ and $\frac{1}{\pi(2k+1)}\int_{\pi 2k}^{\pi(2k+1)} \sin(u)\ du = \frac{1}{\pi(k+1/2)}$. Putting this all together, we see that $\int_{a_k}^{b_k} \sin(e^x)\ dx \in \left(\frac{c}{k+1/2}, \frac{c}{k}\right)$ for $c = 1/\pi$.

We can proceed similarly to conclude that $\int_{b_k}^{a_{k+1}} \sin(e^x)\ dx \in \left(-\frac{c}{k+1/2}, -\frac{c}{k+1}\right)$. It follows that

$$\int_{a_k}^{a_{k+1}} \sin(e^x)\ dx \in \left(0, c\left(\frac{1}{k} - \frac{1}{k+1}\right)\right).$$

By the mean value theorem, $\frac{1}{k} - \frac{1}{k+1} \le C\frac{1}{k^2}$ for some fixed constant C. Therefore

$$\int_{a_k}^{a_{k+1}} \sin(e^x)\ dx \in \left(0, \frac{cC}{k^2}\right).$$

We then see that for $y \in [a_k, a_{k+1}]$,

$$\int_{a_1}^{y} \sin(e^x)\ dx = \sum_{j=1}^{k-1} \int_{a_j}^{a_{j+1}} \sin(e^x)\ dx + \int_{a_k}^{y} \sin(e^x)\ dx \leq cC \sum_{j=1}^{k-1} j^{-2} + \int_{a_k}^{y} \sin(e^x)\ dx.$$

We know that $cC \sum_{j=1}^{k-1} j^{-2}$ converges in k. The estimates we have previously done imply that $\int_{a_k}^{y} \sin(e^x)\ dx \in \left(0, \frac{cC}{k}\right)$, which also converges in k. Since k is an increasing function of y, $\lim_{y\to\infty} \int_{a_1}^{y} \sin(e^x)\ dx$ converges in y as well. Hence, the given integral does converge.

Of course, that is not the integral we were asked to compute. But now we know that $\int_2^\infty \frac{x \sin(e^x)}{x+\sin(e^x)}\ dx$ converges if and only if $\int_2^\infty \frac{x\sin(e^x)}{x+\sin(e^x)} - \sin(e^x)\ dx$ converges, or equivalently, if and only if $\int_2^\infty \frac{\sin^2(e^x)}{x+\sin(e^x)}\ dx$ converges. This is better, because the integrand is strictly non-negative, so if we make the denominator of the integrand bigger, it makes the integral smaller. If $x > 2$, then $\sin(e^x) + x < 2x$, so

$$\int_2^\infty \frac{\sin^2(e^x)}{x+\sin(e^x)}\ dx > \frac{1}{2}\int_2^\infty \frac{\sin^2(e^x)}{x}\ dx.$$

We will prove the final integral diverges, completing the problem. Let's substitute $u = e^x$. The integral becomes $\int_{e^2}^\infty \frac{\sin^2(u)}{u\log(u)}\ du$. At this point. In a small neighborhood $I = (\pi/2 - \varepsilon, \pi/2)$, $\sin^2(u) \geq \frac{1}{2}$. Since $\sin^2(u)$ is 2π-periodic, $\sin^2(u) \geq \frac{1}{2}$ on $I_n = 2\pi n + I$ as well. Then if $a_n = 2\pi n + \pi/2$, we have

$$\int_{e^2}^\infty \frac{\sin^2(u)}{u\log(u)}\ du \geq \sum_{n=0}^\infty \int_{I_n} \frac{\sin^2(u)}{u\log(u)}\ du \geq \sum_{n=0}^\infty \frac{1}{a_n \log(a_n)}.$$

We can prove that $\sum_{n=0}^\infty \frac{1}{a_n\log(a_n)}$ diverges by the integral comparison test: since $\frac{1}{a_n\log(a_n)} \geq \frac{1}{(2\pi x+\pi/2)\log(2\pi x+\pi/2)}$ for $x \geq n$, we know that $\sum_{n=0}^\infty \frac{1}{a_n\log(a_n)} \geq \int_1^\infty \frac{1}{(2\pi x+\pi/2)\log(2\pi x+\pi/2)}\ dx$. By substituting $y = 2\pi x$ and then $z = y + \pi/2$, we see that this converges if and only if $\int_2^\infty \frac{1}{x\log(x)}\ dx$ converges. Let $u = \log(x)$, so $du = \frac{dx}{x}$, and the integral becomes the definitely divergent $\int_{\log(2)}^\infty \frac{1}{u}\ du$.

Exercise 58. Find the spectrum of the linear operator A in $L^2(\mathbb{R})$ defined as

$$(Af)(x) = \int_{-\infty}^{\infty} \frac{f(y)}{1+(x-y)^2}\ dy.$$

(The spectrum of a linear operator T is the closure of the set of all complex numbers λ such that the operator $T - \lambda I$ does not have a bounded inverse. Hint: it may be helpful to find Fourier transform of $1/(1+x^2)$.)

Solution 58. Let's start with the hint. We will prove that the Fourier transform of $\frac{1}{1+x^2}$ is $\pi e^{-|x|}$. It is not too hard to check that this is correct by Fourier inverting $\pi e^{-|x|}$ and splitting up the domain of the resulting integral, but actually coming up with that on your own would be tricky. Here is one approach. Let $g(\xi)$ denote the Fourier transform of $\frac{1}{1+x^2}$. Differentiating under the integral, we see that $\frac{\partial^2}{\partial\xi^2} g(\xi) = -\hat{1} + g(\xi)$. Of course, $\hat{1}$ only makes sense as a distribution, but with a bit of distribution theory and the fact that $\int f(x)\ dx = \check{f}(0)$, we see that it is δ_0: for any $h \in C_c^\infty(\mathbb{R})$, $(\hat{1}, h) = (1, \hat{h}) = \check{\hat{f}}(0) = f(0)$.

Now we have a differential equation, which does something weird at 0, but on $(-\infty,0)$ and $(0,\infty)$, we should be able to solve this as one normally would (which for me means plug in $e^{a\xi}$ and hope a solution falls out). In thise case, we are in luck: we see that any solution must be of the form $Ce^{a\xi}$ for $a=\pm 1$. Since $\frac{1}{1+x^2}\in L^1(\mathbb{R})$ and the Fourier transform of an L^1 function is in $C_0(\mathbb{R})$, we know that we must have $a=1$ on $(-\infty,0)$ and $a=-1$ on $(0,\infty)$ for g to have good decay at ∞. For g to be continuous at 0, we must have the same constant factor throughout, which must equal $g(0)$. It is straightforward to compute that $g(0)=\pi$, and we are left with $g(\xi)=\begin{cases}\pi e^{-\xi} & \xi\geq 0\\ \pi e^{\xi} & \xi<0\end{cases}=\pi e^{-|\xi|}$.

Now that we have that out of the way, we return to the actual problem. Note that $Af=f*h$, where $h(x)=\frac{1}{1+x^2}$. Then $\hat{Af}=\hat{f}\hat{h}=\pi e^{-|x|}\hat{f}$, so $(Af-\lambda f)\hat{}=(\pi e^{-|x|}-\lambda)\hat{f}$. Then if $A-\lambda I$ has an inverse $G_\lambda: L^2\to L^2$, then if $\hat{Gf}$ is well defined, it must equal $\frac{1}{\pi e^{-|x|}-\lambda}\hat{f}$. If $\lambda\notin[0,\pi]$, then by Hölder's inequality and Plancheral, $||\hat{Gf}||_{L^2}\leq\frac{1}{|\lambda|+\pi}||f||_{L^2}$. Then $\hat{Gf}$ is bounded $L^2\to L^2$, so by Plancheral again, Gf is as well. Note that $\hat{Af}=\pi e^{-|x|}\hat{f}$ has non-trivial kernel, since any f with $\hat{f}\perp\pi e^{-|x|}$ will be sent to 0, so Af has the same non-trivial kernel, and hence $0\in\operatorname{spec}(A)$.

Finally, we will check that $(0,\pi]\in\operatorname{spec}(A)$. Take $\lambda\in(0,\pi]$ and choose x_0 such that $\pi e^{-|x_0|}=\lambda$ (there usually will be two such values, choose one). Take $f\in L^2$ such that $\hat{f}=\chi_{[x_0-1,x_0+1]}$. If $\hat{Gf}$ is bounded $L^2\to L^2$, then $\frac{\chi_{[x_0-1,x_0+1]}(x)}{\pi e^{-|x|}-\lambda}\in L^2$, or in other words, $\int_{x_0-1}^{x_0+1}\frac{1}{|\pi e^{-|x|}-\lambda|^2}\,dx<\infty$, which would imply that $|\pi e^{-|x|}-\lambda|\gtrsim|x-x_0|^{1/2}$ as $x\to x_0$. But by the mean value theorem, we know that $|\pi e^{-|x|}-\lambda|=|\pi e^{-|x|}-\pi e^{-|x_0|}|\lesssim|x-x_0|\ll|x-x_0|^{1/2}$ as $x\to x_0$. Therefore, $\hat{Gf}$ cannot be bounded $L^2\to L^2$ and hence neither can Gf. It follows that $\lambda\in\operatorname{spec}A$, and since $\lambda\in(0,\pi]$ was arbitrary, we see that $(0,\pi]\subset\operatorname{spec}(A)$. Putting this all together, we conclude that $\operatorname{spec}(A)=[0,\pi]$.

Exercise 59. Let $\{x_n\}_{n=1}^\infty$ be a sequence of elements in a Hilbert space H. Suppose that $x_n\to x\in H$ weakly in H and that $||x_n||\to||x||$ as $n\to\infty$. Show that then $||x_n-x||\to 0$. Would the same be true for an arbitrary Banach space in place of H?

Solution 59. We can write $||x_n-x||^2=\langle x_n-x,x_n-x\rangle$. Using linearity of the inner product, we see that $\langle x_n-x,x_n-x\rangle=||x_n||^2+||x||^2-2\langle x_n,x\rangle$. Since $x_n\to x$ weakly, we know $\langle x_n,x\rangle\to||x||^2$, so since $||x_n||^2\to||x||^2$ as well, we have $||x_n||^2+||x||^2-2\langle x_n,x\rangle\to 0$. Hence, $||x_n-x||^2\to 0$, so $||x_n-x||\to 0$ as well.

The same is not true for arbitrary Banach spaces. Denote by 1 the sequence $(1,1,1,\dots)\in\ell^\infty$. Define the sequence $x^n=1-e^n\in\ell^\infty(\mathbb{N})$ (that is, $(x^n)_m=1$ for $m\neq n$ and 0 for $m=n$). Clearly, $||x^n||_{\ell^\infty}=1$ for all n. We also have that $||1-x^n||_{\ell^\infty}=1$ for all n. We will prove that x^n converges weakly to 1. It suffices to prove that e^n converges weakly to 0. Suppose otherwise. Then there exists $f\in(\ell^\infty(\mathbb{N}))^*$ such that $f(e^n)\not\to 0$. Then for some $\varepsilon>0$, there exists a subsequence $n_k\to\infty$ such that $|f(e^{n_k})|\geq\varepsilon$ for all k. Define $\lambda\in\ell^\infty$ to be $f(e^{n_k})$ at the n_kth entry for all k and 0 everywhere else. Then $|\lambda_n|\leq||f||$ for all n, so $\lambda\in\ell^\infty$, but $f(\lambda)=\sum_{k=1}^\infty f(e^{n_k})^2\geq\sum_{k=1}^\infty\varepsilon^2=\infty$, a contradiction. Hence, e^n converges weakly to 0, completing the problem.

The second part is very tricky, I think it would quite challenging to come up with a counterexample on the qual if you did not already know it.

Exercise 60.

(1) Prove or disprove: there exists a distribution $u \in \mathcal{D}'(\mathbb{R})$ so that its restriction to $(0, \infty)$ is given by
$$\langle u, f\rangle = \int_0^\infty e^{1/x^2} f(x)\ dx$$
for all C^∞ which are compactly supported in $(0, \infty)$.

(2) Prove or disprove: there exists a distribution $u \in \mathcal{D}'(\mathbb{R})$ so that its restriction to $(0, \infty)$ is given by
$$\langle u, f\rangle = \int_0^\infty x^{-2} e^{i/x^2} f(x)\ dx$$
for all C^∞ which are compactly supported in $(0, \infty)$.

Solution 60.

(1) This is false. Suppose it was true. Then there would exist some N such that $\langle u, f\rangle \lesssim ||f||_{C^N}$ for smooth functions f supported on $[0, 1]$. Let ϕ be a smooth bump function supported on $[1/4, 3/4]$, equal to 1 on $[1/3, 2/3]$, and everywhere non-negative. Define $\phi_\varepsilon = \phi(x/\varepsilon)$. Repeatedly differentiating, we see that $||\phi_\varepsilon||_{C^N} \lesssim \varepsilon^{-N}||\phi||_{C^N}$, so $\varepsilon^N\langle u, \phi_\varepsilon\rangle \lesssim 1$. Changing variables, we see that $\langle u, \phi_\varepsilon\rangle = \varepsilon\langle u_\varepsilon, \phi\rangle$, where $u_\varepsilon = e^{1/(\varepsilon x)^2}$. It suffices then to prove that for any $C > 0$, there exists ε sufficiently small such that $\langle u_\varepsilon, \phi\rangle > C\varepsilon^{-(N+1)}$. Set $\varepsilon = \frac{3}{2\log(A)^{1/2}}$ for A a large number. Then for $x \in [1/3, 2/3]$ $e^{1/(\varepsilon x)^2} \geq A$, so $\langle u_\varepsilon, \phi\rangle \geq \frac{1}{3}A$. But for A sufficiently large, $A > C(2/3 \log(A))^{(N+1)/2}$, since $\lim_{A\to\infty} \frac{A}{\log(A)^{(N+1)/2}} = \infty$. Therefore, we can find an $\varepsilon > 0$ such that $\langle u_\varepsilon, \phi\rangle \gtrsim \varepsilon^{-(N+1)}$, so u cannot be a distribution.

(2) Let's write $\langle u, f\rangle = -2i \int_0^\infty \frac{-e^{i/x^2}}{2ix^3} xf(x)\ dx$. If f is compactly suported on $(0, \infty)$, then integrating parts, we see that
$$\langle u, f\rangle = 2i \int_0^\infty e^{i/x^2}[xf(x)]'\ dx = 2i \int_0^\infty xe^{i/x^2} f'(x)\ dx + 2i \int_0^\infty e^{i/x^2} f(x)\ dx.$$
As both $xe^{i/x^2}, e^{i/x^2} \in L^1_{\text{loc}}$, both are distributions, and hence so is $\langle u, f\rangle$.

Exercise 61. Determine if
$$\sum_{n=1}^\infty \frac{\cos(k)}{k}$$
converges.

Solution 61. I believe this can also be done by a careful application of the integral comparison test, but I will solve it using summation by parts. We will first bound the "integral" terms, which will come from summing $\cos(k)$. Let $C_k = \sum_{j=1}^k \cos(j) = \frac{1}{2}\left(\sum_{j=1}^k e^{ik} + \sum_{j=1}^k e^{-ik}\right)$. By the geometric sum formula, $\left|\sum_{j=1}^k e^{ik}\right| \leq \frac{2}{|1-e^i|}$ and $\left|\sum_{j=1}^k e^{-ik}\right| \leq \frac{2}{|1-e^{-i}|}$, so $|C_k| = \frac{1}{2}\left|\sum_{j=1}^k e^{ik} + \sum_{j=1}^k e^{-ik}\right| \leq C$ for some absolute constant C. The "derivative" terms $\left|\frac{1}{k} - \frac{1}{k+1}\right| = \frac{1}{k(k+1)} \leq \frac{1}{k^2}$. We now see that that the product of the "integral" terms and the "derivative" terms is bounded above by $\frac{C}{k^2}$, a summable sequence. As long as the boundary terms converge (and unless the divergence test fails, one should expect the boundary terms to always converge), we should expect summation by parts to show convergence.

To do this carefully, recall that the summation by parts formula tells us that for $N \geq M$,

$$\left|\sum_{k=M}^{N} \frac{\cos(k)}{k}\right| \leq \frac{|C_N|}{N} + \frac{|C_M|}{M} + \sum_{k=M}^{N-1} \left|\frac{1}{k} - \frac{1}{k+1}\right| |C_k| \leq \frac{2}{M} + C \sum_{k=M}^{N-1} \frac{1}{k^2}.$$

As each term converges to 0 in M (for the final sum, this is a consequence of the fact that $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges), we must have that $\sum_{k=1}^{N} \frac{\cos(k)}{k}$ is a Caucy sequence in N, and hence converges as well.

Exercise 62. For a Lebesgue measurable subset E of $\mathbb{R}$, denote by χ_E the indicator function of E. Let $\{E_n : n \in \mathbb{N}\}$ be a family of Lebesgue measurable subsets of $\mathbb{R}$ with finite measure and let f be a measurable function such that

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f(x) - \chi_{E_n}|\ dx = 0.$$

Prove that f is almost everywhere equal to the indicator function of a measurable set.

Solution 62. I think the most natural thing to do is to prove the sets $L_\varepsilon = \{x : ||f|(x) - 1| > \varepsilon \text{ and } ||f(x)| - 0| > \varepsilon\}$ have measure 0 by computing $\int_{L_\varepsilon} |f(x) - \chi_{E_n}(x)|\ dx$, then conclude that that f must equal 1 or 0 a.e.. I will give a less natural proof, which is quicker but requires a little more machinery.

Suppose χ_{E_n} converges pointwise a.e.. to f. The limit of a pointwise convergent sequence taking values in $\{0, 1\}$ must either by 0 or 1, so f takes values in $\{0, 1\}$ a.e.. Therefore, f a.e. equals $\chi_{f^{-1}(\{1\})}$. It is possible that χ_{E_n} does not converge a.e.., but since it converges in L^1, it has a subsequence E_{n_k} that converges a.e.. The subsequence still satisfies $\lim_{k\to\infty} \int |E_{n_k} - f|\ dx = 0$, so we follow the same proof to conclude that f is a characteristic function.

The problem asks for E to be measurable, which it is because f is measurable (since it is in L^1) and so $f^{-1}(\{1\})$ is measurable (technically, that set is only determined up to sets of measure zero, but sets of measure zero are always Lebesgue measureable).

Exercise 63. For a Lebesgue measurable subset E of $\mathbb{R}$, denote by χ_E the indicator function of E. Let $\{E_n : n \in \mathbb{N}\}$ be a family of Lebesgue measurable subsets of $\mathbb{R}$ with finite measure and let f be a measurable function such that

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f(x) - \chi_{E_n}|\ dx = 0.$$

Prove that f is almost everywhere equal to the indicator function of a measurable set.

You've seen this problem before, but I'd invite you to think about a very short proof using some facts from modes of convergence.

Solution 63. See the solution in the day 6 solution.

Exercise 64. Let f be a C^1 function on $[0, \infty)$. Suppose that

$$\int_0^\infty t|f'(t)|^2\ dt < \infty,$$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(t)\ dt = L.$$

Show that $f(t) \to L$ as $t \to \infty$.

Solution 64. This is kind of like a Cesaro summation problem, except with integrals, which I find makes things easier. Fix $\varepsilon > 0$, let's prove that for S sufficiently large and $t > S$, $|f(t) - L| < \varepsilon$. Since $\int_0^\infty t|f'(t)|^2\ dt < \infty$, for S_0 sufficiently large, $\int_{S_0}^\infty t|f'(t)|^2\ dt < (\varepsilon/3)^2$. Integrating by parts, we see that $\int_{S_0}^T f(t)\ dt = Tf(T) - S_0 f(S_0) - \int_{S_0}^T tf'(t)\ dt$. Therefore, $f(T) = \frac{\int_{S_0}^T f(t)\ dt}{T} + \frac{S_0 f(S_0)}{T} + \frac{\int_{S_0}^T tf'(t)\ dt}{T}$. The last term is the hardest to control, so we will do so first. By Cauchy-Schwartz (applied with respect to the measure $d\mu = t\ dt$), for $T \geq S_0$, we have

$$\int_{S_0}^T f'(t)t\ dt \leq \left(\int_{S_0}^T |f'(t)|^2 t\ dt\right)^{1/2} \left(\int_{S_0}^T t\ dt\right)^{1/2} \leq \frac{\varepsilon}{3}((T^2 - S_0^2)/2)^{1/2} \leq \frac{\varepsilon T}{3}.$$

Then $\frac{\int_{S_0}^T f'(t)t\ dt}{T} \leq \frac{\varepsilon}{3}$.

Clearly, taking T sufficiently large, we can ensure that $\left|\frac{S_0 f(S_0)}{T}\right| \leq \frac{\varepsilon}{3}$. Finally, we want to prove that for T sufficiently large, $\left|\frac{\int_{S_0}^T f(t)\ dt}{T} - L\right| < \frac{\varepsilon}{3}$. Since $\lim_{T\to\infty} \frac{\int_0^T f(t)\ dt - \int_{S_0}^T f(t)\ dt}{T} = \lim_{T\to\infty} \frac{\int_0^{S_0} f(t)\ dt}{T} = 0$, we know that $\lim_{T\to\infty} \frac{\int_{S_0}^T f(t)\ dt}{T} = \lim_{T\to\infty} \frac{\int_0^T f(t)\ dt}{T} = L$, imply that we can find T large enough so that $\left|\frac{\int_{S_0}^T f(t)\ dt}{T} - L\right| < \frac{\varepsilon}{3}$ holds.

Taking all the bounds we have estabilished, we see that $|f(T) - L| \leq \varepsilon$ for T sufficiently large. Since $\varepsilon < 0$ was arbitrary, we see that $\lim_{T\to\infty} f(T) = L$, as desired.

Exercise 65. Let K be a continuous function on $[0,1] \times [0,1]$ satisfying $|K| < 1$. Suppose that g is a continuous function on $[0,1]$. Show that there exists a continuous function f on $[0,1]$ such that

$$f(x) = g(x) + \int_0^1 f(y)K(x,y)\ dy.$$

Solution 65. This is a contraction mapping problem. Our operator will be $Tf(x) = g(x) + \int_0^1 f(y)K(x,y)\ dy$. Note that since K is continuous on a compact interval, it acheives it's supremum. Since $|K| < 1$, we therefore know $|K| \leq c$ for some $c < 1$. Then $|Tf_1(x) - Tf_2(x)| \leq \int_0^1 |f_1(y) - f_2(y)||K(x,y)|\ dy \leq c \sup_{x\in[0,1]} |f_1(x) - f_2(x)|$. It follows that $d(Tf_1, Tf_2) \leq cd(f_1, f_2)$, so T is a contraction mapping. Then by the contraction mapping theorem, it has a fixed point f. Hence, f satisfies $f(x) = g(x) + \int_0^1 f(y)K(x,y)\ dy$.

Exercise 66. For $a, b \geq 0$, let

$$F(a,b) = \int_{-\infty}^\infty \frac{dx}{x^4 + (x-a)^4 + (x-b)^4}.$$

For what values of $p \in (0,\infty)$ is

$$\int_0^1 \int_0^1 F(a,b)^p\ da\ db < \infty?$$

Hint: try to prove that when $a \leq b$, $b^{-3}c \leq F(a,b) \leq b^{-3}C$ for positive constant $c < C$.

Solution 66. Explicitly computing $F(a,b)$ seems rather difficult. Instead, we will attempt to prove the approximate bound in the hint, with the assumption in the hint that $a \le b$. When trying to bound $F(a,b)$, we need to somehow capture both the decay of $g(x) = \frac{1}{x^4+(x-a)^4+(x-b)^4}$ in the denominator at for large values of x and that $g(x)$ is bounded for small values of x. We can approximate $g(x)$ with one of the $(x-c)^{-4}$ terms to get decay when x is large, but we have to be careful in which one we choose to avoid issues when x is small. Or at least, my first couple attempts went nowhere.

We have assumed that $0 \le a \le b$, and without loss of generality, we may further assume that $0 < a < b$, since what is left in the final integral we want to bound is negligible. This allows us to choose our large value approximations $(x-c)^{-4}$ to avoid ever hitting a pole, while being useful over almost the entire domain. We will choose to approximate $g(x)$ with $\frac{1}{(x-a)^4}$ when $x > a+b$ or $x < a-b$ and with $\frac{1}{b^4}$ when $x \in (a-b, a+b)$. Let $h(x)$ the function given by our approximation.

To be rigorous about the approximations, it is easy to see that when $x > a+b$, $(x-a)^4 < x^4 + (x-a)^4 + (x-b)^4$, $(x-a)^4 \ge (x-a)^4$ and $(x-a)^4 \ge (x-b)^4$. We know that $x > a+b > 2a$, so $\frac{-a}{x} \ge \frac{-1}{2}$, and hence $(1-a/x)^4 > \frac{1}{16}$. Therefore, $(x-a)^4 > \frac{x^4}{16}$. Therefore, $(x-a)^4 > C_0(x^4+(x-a)^4+(x-b)^4)$ for some small enough constant C_0. We can conclude by taking the reciprocal of these inequalities that $h(x) \approx g(x)$ for $x > a+b$. For $x < a-b$, we proceed similarly to see that $h(x) \approx g(x)$. For $x \in (a-b, a+b)$, we write $x^4 + (x-a)^4 + (x-b)^4 = (y+a)^4 + y^4 + (y+a-b)^4$ for $y = x-a \in (-b, b)$. The latter polynomial can be expanded to a sum of D (a large absolute constant) monomials of degree 4, made up y, a, or b, and hence can be bounded above by Db^4 and, as one of those terms must necessarily be b^4, it can be bounded below by b^4 itself. It follows that $g(x) \approx h(x)$ in that range as well, and hence that $\frac{1}{C}\int_{-\infty}^{\infty} h(x)\,dx \le \int_{-\infty}^{\infty} g(x)\,dx \le \int_{-\infty}^{\infty} h(x)\,dx$.

For me, validifying the approximations was the most difficult part of the problem, and the rest was smooth sailing. We see that $\int_{a+b}^{\infty} \frac{1}{(x-a)^4}\,dx + \int_{-\infty}^{a-b} \frac{1}{(x-a)^4} = \frac{C_1}{b^3}$ for some absolute constant C_1. We also see that $\int_{a-b}^{a+b} \frac{1}{b^4}\,dx = \frac{2}{b^3}$. Then $\int_{-\infty}^{\infty} g(x)\,dx \in (C_2 b^{-3}, C_3 b^{-3})$ for absolute constants C_2, C_3. I keep saying "absolute constants" to emphasize that they do not depend on a or b, which will be important for what happens next.

We now use this approximation in the integral we actually want to bound. Since $F(a,b) = F(b,a)$, we know that $\int_{[0,1]^2} F(a,b)^p\,da\,db = 2\int_{a<b} F(a,b)^p\,da\,db$. By the approximation we just proved, we see that

$$C_2 \int_{a<b} b^{-3p}\,da\,db \le \int_{a<b} F(a,b)^p\,da\,db \le \int_{a<b} b^{-3p}\,da\,db.$$

Therefore, our desired values of p are precisely those where $\int_{a<b} b^{-3p}\,da\,db < \infty$. Finally, we write $\int_{a<b} b^{-3p}\,da\,db = \int_0^1 b^{1-3p}\,db$, and note by the p test that it converges if and only if $1-3p > -1$, or equivalently, $p < \frac{2}{3}$. This is our final answer.

Exercise 67. Let $f : \mathbb{R} \to \mathbb{R}$ be a compactly supported function that satisfies the Hölder condition with exponent $\beta \in (0,1)$, that is, there exists a constant $A < \infty$ such that for all $x, y \in \mathbb{R}$, $|f(x) - f(y)| \le A|x-y|^\beta$. Consider the function g defined by

$$g(x) = \int_{-\infty}^{\infty} \frac{f(y)}{|x-y|^\alpha}\,dy$$

where $\alpha \in (0,\beta)$

(1) Prove that g is a continuous function at 0.
(2) Prove that g is differentiable at 0. (Hint: Try the dominated convergence theorem.)

Solution 67.

(1) By the substitution $u = x - y$, $g(x) = \int_{-\infty}^{\infty} \frac{f(u-x)}{|u|^\alpha}\, du$. Assume f is supported on $[-M, M]$. Then if $|x| < 1$, and hence $f(u), f(u-y)$ are both supported on $u \in [-M-1, M+1]$, so

$$|g(0) - g(x)| \le \int_{-M-1}^{M+1} \frac{f(u-x) - f(u)}{|u|^\alpha}\, dy < A|x|^\beta \int_{-M-1}^{M+1} \frac{1}{|u|^\alpha}\, du.$$

Since $\alpha < 1$, the integral of $|u|^{-\alpha}$ over $[-M-1, M+1]$ is some finite constant C, so $|g(0) - g(x)| \le AC|x|^\beta$. Hence, g is continuous at 0.

(2) This is an example of what is called a kernel operator. On $\mathbb{R}$, these are operators T associated with a measurable function $K(x,y)$ defined to be $Tf(x) = \int K(x,y) f(y)\, dy$ (in this case, we take $K(x,y) = |x-y|^{-\alpha}$). These operators come up in many guises. A key property of these operators is that they tend to improve the regularity of the input function. Of course, this depends on the properties of K, but in general we should get more regularity in the output than we might expect in the input. In this problem, our kernel has a singularity at 0 and our function isn't differentiable, but the kernel operator outputs a differentiable function. There are various ways we can prove that increase in regularity. When you can figure out the derivative of your kernel, using integration by parts and putting the derivative on the kernel can be an effective way to prove the desired regularity. Using the substitution in the previous problem, we see that

$$\frac{g(x) - g(0)}{x} = \int_{-\infty}^{\infty} \frac{f(u-x) - f(u)}{x|u|^\alpha}\, du.$$

Integrating by parts, we send $u^{-\alpha}$ to $Cu^{-\alpha-1}$ for some constant C, and we send $f(u-x) - f(u)$ to $F(x,y) = \int_0^u f(y-x) - f(y)\, dy$. The boundary terms in the integration by parts vanish, so we see that

$$\frac{g(x) - g(0)}{x} = C \int_{-\infty}^{\infty} \frac{F(x,u)}{x|u|^{\alpha+1}}\, du$$

We can rearrange $F(x,u) = \int_0^{-x} f(y+u) - f(y)\, dy$, so $|F(u)| \le A|x||u|^\beta$. Then $\left|\frac{F(x,u)}{x|u|^{\alpha+1}}\right| \le A\frac{|x||u|^\beta}{|x||u|^{\alpha+1}} = \frac{A}{|u|^{\alpha+1-\beta}}$, which is integrable since $\beta > \alpha$. Then we are justified in using the dominated convergence theorem. By the fundamental theorem of calculus, $\lim_{x\to 0} \frac{F(x,u)}{x} = f(u) - f(0)$, so

$$\lim_{x\to 0} \frac{g(x) - g(0)}{x} = \int_{-\infty}^{\infty} \lim_{x\to 0} \frac{F(x,u)}{u^{\alpha+1}}\, du = \int_{-M}^{M} \frac{f(u) - f(0)}{|u|^{\alpha+1}}\, du,$$

where once again, $[-M, M]$ is the support of f. This final integral is finite because $|f(u) - f(0)| < |u|^\beta$, so $\lim_{x\to 0} \frac{g(x)-g(0)}{x}$ converges, and we are done.

Exercise 68. A real-valued function f defined on $\mathbb{R}$ belongs to the space $C^{1/2}(\mathbb{R})$ if and only if

$$\sup_{x\in\mathbb{R}} |f(x)| + \sup_{x\neq y} \frac{|f(x)-f(y)|}{\sqrt{|x-y|}} < \infty.$$

Prove that a function $f \in C^{1/2}(\mathbb{R})$ if and only if there is a constant C so that for every $\varepsilon > 0$, there is a bounded function $\phi \in C^\infty(\mathbb{R})$ such that

$$\sup_{x\in\mathbb{R}} |f(x)-\phi(x)| \leq C\sqrt{\varepsilon} \text{ and } \sup_{x\in\mathbb{R}} \sqrt{\varepsilon}|\phi'(x)| \leq C.$$

Solution 68. Suppose $f \in C^{1/2}(\mathbb{R})$. Let φ be a smooth bump function, symmetric about the $x = 0$, supported on $[-1, 1]$, with $||\varphi(x)||_{L^1} = 1$. Let $\varphi_\varepsilon(x) = \frac{\varphi(x/\varepsilon)}{\varepsilon}$. Note that φ_ε is supported on $[-\varepsilon, \varepsilon]$ and $||\varphi_\varepsilon||_{L^1} = 1$. Define $\phi = f * \varphi_\varepsilon$. Let's first check that $\sup_{x\in\mathbb{R}} |f(x) - \phi(x)| \leq C\sqrt{\varepsilon}$. We see that

$$|f(x)-\phi(x)| \leq \int_{\mathbb{R}} \varphi_\varepsilon(y)|f(x)-f(x-y)|\, dy \leq \int_{-\varepsilon}^{\varepsilon} \varphi_\varepsilon(y)\sqrt{y}\, dy.$$

By Hölder's inequality, $\int_{-\varepsilon}^{\varepsilon} \varphi_\varepsilon(y)\sqrt{y}\, dy \leq ||\varphi_\varepsilon||_{L^1}||\sqrt{y}||_{L^\infty([-\varepsilon,\varepsilon])} \leq \sqrt{\varepsilon}$.

For the second part, we have $\phi'(x) = \frac{1}{\varepsilon^2}\int_{-\infty}^{\infty} f(x-y)\varphi'(y/\varepsilon)\, dy$. We know that by the assumption that φ is symmetric that about $x = 0$ that φ' is odd, so we can write this as $\frac{1}{\varepsilon^2}\int_0^\infty [f(x-y) - f(x+y)]\varphi'(y/\varepsilon)\, dy$. We know $\varphi'(y)$ is supported on $[-\varepsilon, \varepsilon]$, so $|f(x-y) - f(x+y)| \leq \sqrt{2y} \leq C_1\sqrt{\varepsilon}$ for some $C_1 > 0$. We also have that $\varphi'(y)$ is uniformly bounded by some $C_2 > 0$. Then

$$\frac{1}{\varepsilon^2}\int_0^\infty [f(x-y)-f(x+y)]\varphi'(y/\varepsilon)\, dy \leq \frac{1}{\varepsilon^2}\int_0^\varepsilon C_1\sqrt{\varepsilon}C_2\, dy = \frac{C_1C_2}{\sqrt{e}}.$$

Hence, for $C = C_1C_2$ $|\phi'(x)| \leq \frac{C}{\sqrt{\varepsilon}}$, as desired.

Now suppose that there is a constant C so that for any $\varepsilon > 0$, we can find $\phi \in C^\infty(\mathbb{R})$ such that $\sup_{x\in\mathbb{R}} |f(x) - \phi(x)| \leq C\sqrt{\varepsilon}$ and $\sup_{x\in\mathbb{R}} \sqrt{\varepsilon}|\phi'(x)| \leq C$. First, let's prove that $\sup_{x\in\mathbb{R}} |f(x)|$ is bounded. Take $\phi \in C^\infty(\mathbb{R})$ with maximum M such that $\sup_{x\in\mathbb{R}} |f(x) - \phi(x)| < C$. Then by the triangle inequality, $\sup_{x\in\mathbb{R}} |f(x)| < C + M$. Now, let's prove $\sup_{x\neq y} \frac{|f(x)-f(y)|}{\sqrt{|x-y|}} < \infty$. Take $x \neq y \in \mathbb{R}$, set $\varepsilon = x - y$, and choose ϕ corresponding to that value of ε. Then

$$\frac{|f(x)-f(y)|}{|x-y|} < \frac{|f(x)-\phi(x)|}{\sqrt{|x-y|}} + \frac{|\phi(x)-\phi(y)|}{\sqrt{|x-y|}} + \frac{|\phi(y)-f(y)|}{\sqrt{|x-y|}} \leq C\frac{\varepsilon}{|x-y|} + \frac{C|x-y|}{\sqrt{\varepsilon}|x-y|} + C\frac{\varepsilon}{|x-y|} = 3C.$$

Hence, $\sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|} < \infty$, so $f \in C^{1/2}(\mathbb{R})$.

Exercise 69. Let $\sum_{n=1}^\infty a_n$ be a convergent series. Let $b_n \in \mathbb{R}$ be an increasing sequence with $\lim_{n\to\infty} b_n = \infty$. Show that

$$\lim_{n\to\infty} \frac{1}{b_n}\sum_{k=1}^n b_k a_k = 0.$$

Solution 69. This would be much easier if $\sum_{n=1}^{\infty} a_n$ converged absolutely. If this was the case, we could reach the desired conclusion easily using dominated convergence. But we do not have absolutely convergence, so we have to workd a little harder.

The summation by parts formula tells us that $\sum_{k=1}^{n} b_k a_k = b_n A_n - \sum_{k=1}^{n-1}(b_{k+1} - b_k)A_k$, where $A_k = \sum_{j=1}^{k} a_j$. Let $L = \sum_{n=1}^{\infty} a_n$. Fix $\varepsilon > 0$ and choose N sufficiently large so that $|A_n - L| < \varepsilon/4$ for all $n \geq N$. Choose $M \geq N$ sufficiently large so that for $m \geq M$, $\left|\sum_{k=1}^{N} \frac{b_{k+1}-b_k}{b_m} A_k\right| < \varepsilon/4$ and $\frac{b_{N+1}}{b_m} < \varepsilon/(4L)$. Then for $m \geq M$

$$\begin{aligned}\left|\frac{1}{b_m}\left(\sum_{k=1}^{m} b_k a_k\right)\right| &= \left|A_m - \sum_{k=1}^{N} \frac{b_{k+1} - b_k}{b_m} A_k - \sum_{k=N+1}^{m} \frac{b_{k+1} - b_k}{b_m} A_k\right| \\ &\leq \left|\sum_{k=1}^{N} \frac{b_{k+1} - b_k}{b_m} A_k\right| + \frac{\varepsilon}{4}\left|\sum_{k=N+1}^{m} \frac{b_{k+1} - b_k}{b_m}\right| + \left|A_m - \sum_{k=N+1}^{m} \frac{b_{k+1} - b_k}{b_m} L\right| \\ &\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4}\left|\frac{b_{m+1} - b_{N+1}}{b_m}\right| + \left|L\left(1 - \frac{b_m - b_{N+1}}{b_m}\right)\right| + |A_m - L| \\ &\leq \frac{3\varepsilon}{4} + \frac{Lb_{N+1}}{b_m} \leq \varepsilon\end{aligned}$$

Since ε was arbitrary, we arrive at the desired conclusion.

Exercise 70. Let $L : [0, 1] \to [0, 1]$ be a function satisfying

$$|L(x_2) - L(x_1)| \leq |x_2 - x_1|/4, |L(1/2) - 1/2| < 1/4.$$

Prove that there is a continuous function $f : [0, 1] \to [0, 1]$ satisfying

$$f(x) = (1 - x)L(f(x)) + 1/100.$$

Solution 70. We are looking for a fixed point of a functional $Tf(x) = (1-x)L(f(x)) + \frac{1}{100}$ acting on the complete metric space M of continuous functions from $[0, 1]$ to $[0, 1]$, with sup metric. First, we need to check that is well-defined, that is, that $T(f) \in M$ for any $f \in M$. Since $|L(1/2) - 1/2| < 1/4$, we know that $L(1/2) \in (1/4, 3/4)$. Then for any $x \in [0, 1]$, $|x - 1/2| < 1/2$, so $|L(x) - L(1/2)| \leq 1/8$, and hence $L(x) \in [1/8, 7/8]$. Then if $x \in [0, 1]$, $(1 - x)L(f(x)) \in [0, 7/8]$, so $(1 - x)L(f(x)) + 1/100 \in [0, 1]$. Hence, $Tf(x) \in [0, 1]$ for all x, so $Tf \in M$, and $T : M \to M$ is a well-defined map.

Now, let's check that T is a contraction. For any $f, g \in C([0, 1])$ and point x, $|Tf(x) - Tg(x)| \leq |1 - x||L(f(x)) - L(g(x))| \leq |f(x) - g(x)|/4$. Hence, $\sup_{x\in[0,1]} |Tf(x) - Tg(x)| \leq \frac{1}{4}\sup_{x\in[0,1]} |f(x) - g(x)|$, so $d(Tf, Tg) \leq \frac{1}{4}d(f, g)$. It follows that T has a fixed point $f \in M$, as desired.

Exercise 71. Show that $\int_0^{\infty} \frac{\sin(x)}{x^{2/3}}\, dx$ converges. Determine if

$$\int_1^{\infty} \frac{\sin(x)}{x^{2/3} + \sin(x)}\, dx$$

converges. *Hint: Use Taylor expansion.*

Solution 71. Let's first show that $\int_0^\infty \frac{\sin(x)}{x^{2/3}}\ dx$ converges. There are two ways this integral could diverge: it could grow to quickly at 0 or shrink to slowly at ∞. The first case is easy to rule out: $\sin(x) \le x$, so $\int_0^1 \frac{\sin(x)}{x^{2/3}}\ dx \le \int_0^1 x^{1/3}\ dx < \infty$. The second is not much harder to rule out. We will integrate by parts to conclude $\int_1^\infty \frac{\sin(x)}{x^{2/3}}\ dx = \cos(1) - \frac{2}{3}\int_1^\infty \frac{\cos(x)}{x^{5/3}}\ dx$. We can bound the absolute value of the final integral by $\int_1^\infty \frac{1}{x^{5/3}}\ dx$, which is finite. It follows that $\int_1^\infty \frac{\sin(x)}{x^{2/3}}\ dx$ converges as well.

The second integral converges as well. Once gain, we only need worry about divergence at 0 or at ∞. When $x \in [0,1]$, $\sin(x) \in (x/2, x)$ (you could conclude this by thinking about the Taylor series of sin, thinking about it's derivatives by themselves, or just by thinking about what the graph of sin looks like), so $\frac{\sin(x)}{x^{2/3}+\sin(x)} \le \frac{x}{x^{2/3}+x/2}$. For $x \in [0,1]$, $x^{2/3} \ge x$, so $\frac{x}{x^{2/3}+x/2} \le \frac{x}{3x/2} = \frac{2}{3}$. Hence, $\int_0^1 \frac{\sin(x)}{x^{2/3}+\sin(x)}\ dx \le \frac{2}{3}$. Again, no divergence.

Avoiding divergence at ∞ is a little tricker. Since $\int_1^\infty \frac{\sin(x)}{x^{2/3}}$ converges, it suffices to prove $\int_1^\infty \frac{\sin(x)}{x^{2/3}+\sin(x)} - \frac{\sin(x)}{x^{2/3}}\ dx$ converges. The integrand can be rearranged to $\frac{-\sin^2(x)}{x^{2/3}(x^{2/3}+\sin(x))}$. Now, we are in a better situation, since the integrand is strictly non-positive. It suffices to prove that $\int_1^\infty \frac{\sin^2(x)}{x^{2/3}(x^{2/3}+\sin(x))}\ dx$ converges. But for $x \ge 1$ $\frac{1}{x^{2/3}(x^{2/3}+\sin(x))} \le \frac{2}{x^{4/3}}$, so

$$\int_1^\infty \frac{\sin^2(x)}{x^{2/3}(x^{2/3}+\sin(x))}\ dx \le 2\int_1^\infty \frac{\sin^2(x)}{x^{4/3}}\ dx \le 2\int_1^\infty \frac{1}{x^{4/3}}\ dx.$$

The final integral converges, and we are done.

Exercise 72. Let $E \subset [0,1]$ be a measurable set with positive Lebesgue measure. Morever, assume it satisfied the following property: as long as x and y belong to E, we know $\frac{x+y}{2}$ belongs to E. Prove that E is an interval.

Solution 72. Let $a = \inf E, b = \sup E$. Our aim will be to prove that $(a,b) \subset E$, which implies E is an interval.

First, let's prove that E contains a dense subset of (a,b). Let $a_n, b_n \in E$ be a sequence of elements with $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. We will prove that E contains a dense subset D_n of $[a_n, b_n]$ for any n. By taking the union over D_n, we arrive at a dense subset of (a,b). By inductively bisecting $[a_n, b_n]$ using the property that if $a, b \in E$, then $\frac{a+b}{2} \in E$, we see that

$$D_n = \bigcup_{m=1}^\infty \left\{\frac{ja_n + (2^m - j)b_n}{2^m} : j = 0, 1, \ldots, 2^m\right\} \subset E$$

Now, let's prove that E contains an interval. This is a consequence of the standard result that the convolution of two functions in L^2 is continuous (see the solution to Exercise 17 for a careful proof of this). Since E has finite measure, $\chi_{E/2}$ is in L^2, so $\chi_{E/2} * \chi_{E/2}$ is continuous.

We can check that $\chi_{E/2} * \chi_{E/2}$ is non-zero, because

$$\begin{aligned}\int \chi_{E/2} * \chi_{E/2}(y)\ dy &= \int \int \chi_{E/2}(x)\chi_{E/2}(y-x)\ dx\ dy \\ &= \int \chi_{E/2}(x) \int \chi_{E/2}(y-x)\ dy\ dx \\ &= |E/2| \int \chi_{E/2}(x)\ dx \\ &= |E/2|^2 > 0.\end{aligned}$$

Then we can find x where $\chi_{E/2} * \chi_{E/2}(x) > 0$, which, since $\chi_{E/2} * \chi_{E/2}$ is continuous, implies $\chi_{E/2} * \chi_{E/2}(y) > 0$ for y in some interval I containing x. If $\chi_{E/2} * \chi_{E/2}(y) > 0$, then there exists some z such that $\chi_{E/2}(z) > 0$ and $\chi_{E/2}(y-z) > 0$. It follows that $z, y-z \in \frac{E}{2}$, so since $\frac{E}{2} + \frac{E}{2} \subset E$, $y = z + (y-z) \in E$. Hence, $I \subset E$, as desired.

The final step is to prove that the longest interval in E has length (a,b). The geometric idea is if the longest interval I in E has length less than $b-a$, we can take some point c outside of I but very close to I, and look at $J = I \cup \frac{c+I}{2}$. Since $I \subset E$ and $c \in E$, $J \subset E$. If c is very close to I, then J will be an interval of length greater than I, contradicting the maximality of E. It follows that longest interval in E has length (a,b). To do this carefully, we need to carefully fill in the missing details.

Finally, let $m = \sup_{I \subset E} |I|$, where the supremum is over intervals I contained in E, and $|I|$ denotes the length of I. Since E contains an interval, this is well-defined. We will first prove that m achieves the supremum, that is, there exists an interval $I \subset E$ such that $m = |I|$. Otherwise, we have a sequence of intervals I_j with $\lim_{j\to\infty} |I_j| = m$. Write $I_j = (a_j, b_j)$. By compactness, we can restrict to a subsequence so that (as ordered pairs) $(a_{j_k}, b_{j_k}) \to (a', b')$, and since $\lim_{k\to\infty} b_{j_k} - a_{j_k} = m$, $b' - a' = m$. Then (as intervals), $(a', b') = \bigcup_{k\in\infty} I_{j_k} \subset E$, so the supremum is achieved by (a', b').

Now suppose $m < b - a$. Let's derive a contradiction. Let $I = (a', b') \subset E$ be an interval of length m. We know either $a < a'$ or $b > b'$, since otherwise $|I| = m \geq b - a$. Without loss of generality, assume $a < a'$. Since E contains a dense subset of (a,b), we can find $\varepsilon < \frac{m}{100}$ such that $c = a' - \varepsilon \in E$. Now consider $J = \frac{I+c}{2} \cup I$. We know that $a' + \frac{m}{100} \in I$ and $a' + \frac{m}{100} = \frac{a'-\varepsilon+a'+\varepsilon+\frac{m}{50}}{2} \in \frac{I+c}{2}$, so since $\frac{I+c}{2}$ and I are interesecting intervals, J is an interval. We also now that $a' - \frac{\varepsilon}{4} = \frac{a'-\varepsilon+a'+\frac{\varepsilon}{2}}{2} \in \frac{I+c}{2}$, so $|J| \geq m + \frac{\varepsilon}{4}$, contradicting the maximality of I. Hence, $m = (b,a)$, so E is an interval.

Exercise 73. Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of Lebesgue measurable functions such that f_n converges to f a.e. on $[0,1]$ and such that $||f_n||_{L^2[0,1]} \leq 1$ for all n. Show that

$$\lim_{n\to\infty} ||f_n - f||_{L^1([0,1])} = 0.$$

Solution 73. Fix $\varepsilon > 0$. By Egerov's theorem, we can find a set A of measure $\leq \varepsilon$ where f_n converges uniformly on A^c. Then choosing n sufficiently large, we can ensure that $\sup_{x\in A} |f_n - f|(x) \leq \varepsilon$, so $\int_0^1 |f_n - f|(x)\ dx \leq \varepsilon$. On the other hand, by Hölder's inequality, $\int_{x\in A^c} |f_n - f|(x)\ dx \leq m(A^c)^{1/2} ||f_n - f||_{L^2} \leq \varepsilon^{1/2}(||f_n||_{L^2} + ||f||_{L^2})$. Let's check that $||f||_{L^2} \leq 1$. By Fatou's lemma and the pointwise a.e. convergence, $\int |f|^2(x)\ dx \leq \liminf_{n\in\mathbb{N}} \int |f_n|^2(x)\ dx \leq 1$, as desired. Then plugging this back in, we see that $\int_{x\in A^c} |f_n -$

$f|(x)\,dx \leq 2\varepsilon^{1/2}$, and hence for n sufficiently large, $||f_n - f||_{L^1} \leq \varepsilon + 2\varepsilon^{1/2}$. Taking $\varepsilon \to 0$, we see that $\lim_{n\to\infty} ||f_n - f||_{L^1([0,1])} = 0$.

Exercise 74. A Hamel basis for a vector space X is a collection $\mathcal{H} \subset X$ of vectors such that each $x \in X$ can be written uniquely as a finite linear combination of elements in $\mathcal{H}$. Prove that an infinite dimensional Banach space cannot have a countable Hamel basis. *Hint: Otherwise the Banach space would be first category in itself.*

Solution 74. Suppose X has a countable Hamel basis $\mathcal{H} = x_1, x_2, \ldots$. Define $E_n = \text{span}\{x_1, \ldots, x_n\}$.. Let's prove that E_n is closed and nowhere dense. To see it is nowhere dense, let $B(x,\varepsilon)$ be an open ball in X. If $B(x,\varepsilon) \subset E_n$, then in particular, $x \in E$, so $B(0,\varepsilon) = B(x,\varepsilon) - x \subset E_n$. But if a vector space contains an open ball at the origin, then since vector spaces are closed under scaling, it contains any open ball centered at the origin, and hence $E_n \supset X$. But this would imply X is finite dimensional, a contradiction. Therefore, we conclude that E_n must be nowhere dense.

Let's now prove that E_n is closed. You could probably use the fact that finite dimensional subspaces are closed without proof, but it is not hard to prove, so I will do so. Let $\alpha^m = \sum_{i=1}^n \alpha_i^m x_i$ define a convergent sequence in E_n with limit α. Then it is a Cauchy sequence in X, and hence a Cauchy sequence in E_n (where we equip E_n with the same norm as X). But any finite dimensional vector space over $\mathbb{R}$ is complete, so $\alpha^m \to \beta \in E_n$. Since limits in X are unique, it follows that $\beta = \alpha \in E_n$, and hence E_n is closed.

Now, the Baire category theorem tells us that $\bigcup_{n\in\mathbb{N}} E_n$ is nowhere dense, and hence $\bigcup_{n\in\mathbb{N}} E_n \neq X$. Any linear combination of a finite subset $x_{a_1}, x_{a_2}, \ldots, x_{a_k} \in \mathcal{H}$ is contained in $E_{\max\{a_1,\ldots,a_k\}}$, so the set of finite linear combinations of elements of $\mathcal{H}$ is a subset of E and hence is not all of X either.

Exercise 75.

(1) Suppose Λ is a distribution on $\mathbb{R}^n$ such that $\text{supp}(\Lambda) = \{0\}$. If $f \in C_c^\infty(R^n)$ satisfies $f(0) = 0$, does it follow that the product $f\Lambda = 0$ as a distribution?

(2) Suppose Λ is a distribution on $\mathbb{R}^n$ such that $\text{supp}(\Lambda) \subset K$, where $K = \{x \in \mathbb{R}^n : |x| \leq 1\}$. If $f \in C_c^\infty(\mathbb{R})$ vanishes on K, does it follow that $f\Lambda = 0$ as a distribution?

Solution 75.

(1) No. Consider $\Lambda = \partial_0'$ and $\varphi = x\psi$ for some $\psi \in C_c(\mathbb{R})$ with $\psi(0) \neq 0$. Then $\langle \Lambda, \varphi \rangle = \psi(0) \neq 0$, but $\varphi(0) = 0$.

(2) It does, although the proof is somewhat painful. I'm not sure if you could take certain steps for granted (in particular, the fact that if f vanishes to all orders on K, then $\langle \Lambda, f \rangle = 0$, which would make things much easier). As writing things carefully in Latex takes a long time, I will leave some details for you to sort out, but what I am doing is modifying the of Theorem 6.25 in Rudin's book *Functional Analysis*. You can email me if you have questions.

First, let's prove that if f vanishes on K, then $f^\alpha(x) = 0$ for any $x \in K$ and any multiindex α (since we are on $\mathbb{R}^n$, we have to worry about derivatives in different directions). We will induct on $|\alpha|$. The base case follows from our assumption that f vanishes on K. Now suppose that f that $f^\alpha(x) = 0$ for any α of order n and let β be a multi-index of order $n+1$. Suppose $f^\beta(x) = c > 0$ (the $c < 0$ case is analogous) for some $x \in K$. Since f is smooth, there is an open set U around x such that $f^\beta(y) > c/2$ for $y \in U$. Then $K \cap U$ contains an open set W on which $f^\beta(y) > c/2$.

Since $|\beta| \geq 1$, there is an index i for which $\beta_i \neq 0$. Then $f^\beta = \partial_i f^{\beta'}$ where β' is the same as β except with the ith multiindex decremented by 1. By the induction hypothesis, we know $f^{\beta'}(x) = 0$ for $x \in K$ and hence for $x \in W$, but we can find $y, y + \varepsilon e_i \in W$, in which case $|f(y) - f(y + \varepsilon e_i)| \geq \frac{c\varepsilon}{2}$, a contradiction. Hence, f^β vanishes on K.

Now suppose f vanishes on K. Then, as previously noted, f^α vanishes on K for any multiindex α. Without loss of generality, we may assume f is supported on $\overline{B}(0,2)$, as $\langle \Lambda, g\rangle = 0$ for any g compactly supported on K^c. By the definition of distributions, there exists $N \in \mathbb{N}$ such that $|\langle \Lambda, \varphi\rangle| \leq ||\varphi||_N$ for φ supported on $\overline{B}(0,2)$. Fix $\eta > 0$, we will aim to prove that $|\langle \Lambda, f\rangle| \leq \eta$. Since f vanishes to all orders at K, we know that for ε_η sufficiently small $|x| < 1 + \varepsilon_\eta$, $|D^\alpha f(x)| \leq \eta$ for all $|\alpha| = N$. It follows, by the mean-value theorem, that $|D^\beta f(x)| \leq C_\beta \eta (|x| - 1)^{|\alpha| - |\beta|}$ for fixed constants C_β. We can find a family of bump functions ψ_ε equal to 1 on $\overline{B}(0, 1+\varepsilon)$ and supported on $\overline{B}(0, 1+2\varepsilon)$ such that $|D^\alpha \psi_\varepsilon|(x) \leq C_\alpha \varepsilon^{-|\alpha|}$ for all multiindices α and $\varepsilon > 0$. Then $\psi_\varepsilon \Lambda = \Lambda$, since Λ is supported on K and $\psi_\varepsilon = 1$ on an open neighborhood of K. It follows that

$$\langle \Lambda, f\rangle = \langle \Lambda, \psi_\varepsilon f\rangle \leq \sup_{|\alpha| \leq N} \sup_{1 \leq |x| \leq 1+2\varepsilon} |D^\alpha(\psi_\varepsilon f)(x)|.$$

By the chain rule, $D^\alpha(\psi_\varepsilon f)(x) = \sum_{\beta+\gamma=\alpha} \psi_\varepsilon^\beta(x) f^\gamma(x)$, so if $2\varepsilon < \varepsilon_\eta$, we have

$$\begin{aligned}
|D^\alpha(\psi_\varepsilon f)(x)| \leq \sum_{\beta+\gamma=\alpha} |D^\beta \psi_\varepsilon|(x) |D^\gamma f|(x) &\leq \sum_{\beta+\gamma=\alpha} C_\alpha C_\beta \eta (|x|-1)^{|\alpha|-|\gamma|} \varepsilon^{-|\beta|} \\
&\leq 2 \sum_{\beta+\gamma=\alpha} C_\alpha C_\beta \eta \varepsilon^{|\alpha|-|\beta|-|\gamma|} \\
&= C\eta.
\end{aligned}$$

All the constants denote fixed numbers (in particular, they do not depend on η), so since η was arbitrar,y we have that $\langle \Lambda, f\rangle = 0$, as desired.

Exercise 76. Show that $\ell^1(\mathbb{N}) \subsetneq (\ell^\infty)^*(\mathbb{N})$. *Hint*: Consider the sequence of averages

$$\phi_n(x) = \frac{1}{n} \sum_{j=1}^n x_j, \quad x = (x_1, x_2, \dots) \in \ell^\infty(\mathbb{N}).$$

Show that $\phi_n \in (\ell^\infty(\mathbb{N}))^*$ and consider its weak-* limit points.

Solution 76. Since ϕ_n is a linear combination of the entries of x, it is linear. To see that it is bounded, note that for any $x \in \ell^\infty$,

$$|\phi_n(x)| \leq \frac{1}{n} \sum_{j=1}^n |x_j| \leq \frac{1}{n} \sum_{j=1}^n ||x||_{\ell^\infty} = ||x||_{\ell^\infty}.$$

By Banach-Alaoglu, it has weak-$*$ limit points. Let ϕ be one of them. We now have to find some way of proving that $\phi \notin \ell^1(\mathbb{N})$. Suppose $\phi \in \ell^1(\mathbb{N})$. Let $e^n \in \ell^\infty(\mathbb{N})$ be 1 at the nth entry and 0 elsewhere. Let $E^n = \sum_{k=1}^n e^n$. Then $\phi(E^n) = \sum_{k=1}^n \phi_k$ (we've assumed that $\phi \in \ell^1(\mathbb{N})$, so this makes sense). Since ϕ is the weak limit of ϕ_{n_k}, $\phi(E^n) = \lim_{k\to\infty} \frac{n}{n_k} = 0$ for any n, so $\sum_{k=1}^n \phi_k = 0$ for any n, and hence $\phi_n = 0$ for any n, so $\phi = 0$. Therefore,

$\phi(1) = 0$, but since $\phi_n(1) = 1$ for all n, this is a contradiction. Therefore, $\phi \notin \ell^1(\mathbb{N})$, so $\ell^1(\mathbb{N}) \subsetneq (\ell^\infty)^*(\mathbb{N})$, as desired.

Exercise 77. Let $K \subset \mathbb{R}^d$ be compact and let μ be a regular Borel measure on K with $\mu(K) = 1$. Prove that there exists a compact set $K_0 \subset K$ such that $\mu(K_0) = 1$ but $\mu(H) < 1$ for every compact $H \subsetneq K_0$.

Solution 77. Let $K_0 = \bigcap K'$, where the intersection is taken over all compact subsets K' of K with $\mu(K') = 1$. The intersection of compact sets is always compact, so K_0 is compact. Moreover, if $H \subsetneq K_0$ satisfies $\mu(H) = 1$, then H is in the family of compact sets with measure 1, and hence $K_0 \subset H$, a contradiction. Hence, $\mu(H) < 1$ for every compact $H \subset K_0$. It remains to prove that $\mu(K_0) = 1$. Clearly, $\mu(K_0) \leq 1$, so suppose $\mu(K_0) < 1$. Since μ is regular, $\mu(K_0) = \inf\{\mu(U) : K \subset U, U \text{ open}\}$. Then if $\mu(K_0) < 1$, there exists $U \subset K$ with $K_0 \subset U$ and $\mu(U) < 1$. Since $K_0 \subset U$, $K \setminus U \subset K \setminus K_0 = \bigcup U'$, where the union is taken over all open sets with measure 0. We can assume this union is countable by noting that $\bigcup U' = \bigcup V'$, where the union is taken over all basis elements with measure 0, and noting that there are only countably many basis elements. It follows that $\mu\left(\bigcup V'\right) \leq \sum \mu(V') = 0$, so since $\mu\left(K \setminus U\right) \leq \mu\left(K \setminus K_0\right)$, $\mu\left(K \setminus U\right) = 0$, so $\mu(U) = 1$, contradicting our earlier assumption that $\mu(U) < 1$.

AN: I do not believe the condition that μ is regular is necessary for this problem.

Exercise 78. Assume that for every $x \in (0, 1)$, the function f is absolutely continuous on $[0, x]$ and bounded variation on $[x, 1]$. Assume also that f is continuous at 1. Prove that f is absolutely continuous on $[0, 1]$.

Solution 78. A function f is absolutely continuous if and only if it is BV, continuous, and maps measure zero sets to measure zero sets. The function f is clearly BV, it is continuous at 1 by assumption and at any $x < 1$ because it is absolutely continuous on $[0, \frac{1+x}{2}]$. If N is a measure zero set, then $|f(N)| = \lim_{x\to 1} |f(N \cap [0, x])| = 0$, since f is absolutely continuous on $[0, x]$, so it maps measure zero subsets of $[0, x]$ to measure zero sets and hence $f(N \cap [0, x]) = 0$ for all x.

Exercise 79. Let $H^1([0, 1]) = \{f \in L^2([0, 1]) : f' \in L^2\}$, where f' denotes the distributional derivative of f. Equip H^1 with the norm $||f||_{H^1} = ||f||_{L^2} + ||f'||_{L^2}$.

For $\alpha \in [0, 1]$, denote $||f||_{C^\alpha} = \sup_{x\in[0,1]} |f(x)| + \sup_{x\neq y\in[0,1]} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$ and $C^\alpha([0, 1]) = \{f \in C([0, 1]) : ||f||_{C^\alpha} < \infty\}$.

You may use without proof that H^1 and C^α are both Banach spaces.

(1) Prove that $H^1([0, 1]) \subset C^{1/2}([0, 1])$.
(2) Prove that the closed unit ball in $H^1([0, 1])$ is compact in $C^\alpha([0, 1])$ for any $\alpha < 1/2$.
(3) Is the closed unit ball in $H^1([0, 1])$ compact in $C^{1/2}([0, 1])$? Prove or give a counterexample.

Solution 79.

(1) Take $f \in H^1([0, 1])$. Formally, we want to say that $|f(y) - f(x)| = |\langle f, \delta_y - \delta_x\rangle| = |\langle f', \chi_{[x,y]}\rangle| \leq ||f'||_{L^2}|x - y|^{1/2} \leq ||f||_{H^1}|x - y|^{1/2}$. For $f \in C^\infty([0, 1]) \subset H^1([0, 1])$, we see that this is true just using standard facts about distributions (in particular, that $\chi'_{[x,y]} = \delta_y - \delta_x$, which follows from the fundamental theorem of calculus). Smooth functions are dense in $H^1([0, 1])$ (I would suggest on the qual assuming that smooth

functions are dense in every function space, then going back and proving that is the case if you have time. In this case, it follows fairly easily by an approximation of the identity argument, you can google it if you are curious), so $f \in H_1$, let $\varphi_n \in C^\infty([0,1])$ be a sequence with $\varphi_n \to f$ in the H^1 norm. It follows that $|(\varphi_n - f)(y) - (\varphi_n - f)(x)| \leq ||\varphi_n - f||_{H^1}|x-y|^{1/2}$. By the triangle inequality, we see that

$$\begin{aligned} |f(y) - f(x)| &\leq |(\varphi_n - f)(y) - (\varphi_n - f)(x)| + |\varphi_n(y) - \varphi_n(x)| \\ &\leq (||\varphi_n - f||_{H^1} + ||\varphi_n||_{H^1})|x-y|^{1/2} \\ &\to ||f||_{H^1}|x-y|^{1/2} \end{aligned}$$

Hence $|f(y) - f(x)| \leq ||f||_{H^1}|x-y|^{1/2}$, so $\sup_{x \neq y \in [0,1]} \frac{|f(x)-f(y)|}{|x-y|^{1/2}} \leq ||f||_{H^1}$. We also see that f is necessarily continuous, a nice thing not to have to also check and justifying considering the pointwise behavior of f.

Now suppose $\sup_{x \in [0,1]} |f(x)| > 2||f||_{H^1}$. Then for some x_0, $f(x_0) > 2||f||_{H^1}$. We know that $|f(x_0) - f(y)| \leq ||f||_{H^1}|x-y|^{1/2}$, so $|f(y)| > 2||f||_{H^1} - ||f||_{H^1}|x_0 - y|^{1/2} \geq ||f||_{H^1}$. Therefore, $\int_0^1 |f(x)|^2\,dx > ||f||_{H^1}^2 \geq ||f||_{L^2}^2 = \int_0^1 |f(x)|^2\,dx$, a contradiction. Hence, $\sup_{x\in[0,1]} |f(x)| < 2||f||_{H^1}$, so putting this together, we have $f \in C^{1/2}([0,1])$, with $||f||_{C^{1/2}([0,1])} \leq C||f||_{H^1}$ for a constant $C > 0$.

(2) By the previous problem, we know that the unit ball in H^1 is contained in the unit ball in $C^{1/2}$. We will prove that the $C^{1/2}$ unit ball B is compact in $C^\alpha([0,1])$ for any $\alpha < 1/2$. Take $f_n \in B$. By Arzela-Ascoli (using the fact that $f_n \in B$ to obtain both uniform boundedness and equicontinuity), f_n has a uniformly convergent subsequence $f_{n_k} \to g$. Since f_{n_k} converges uniformly, it is Cauchy in the sup-norm. We also know that

$$\begin{aligned} \sup_{x,y\in(0,1)} \frac{|(f_{n_k} - f_{n_j})(x) - (f_{n_k} - f_{n_j})(y)|}{|x-y|^\alpha} &= \sup_{x,y\in(0,1)} \left(\frac{|(f_{n_k} - f_{n_j})(x) - (f_{n_k} - f_{n_j})(y)|^{1/(2\alpha)}}{|x-y|^{1/2}} \right)^{2\alpha} \\ &= \sup_{x,y\in(0,1)} \left(\frac{|(f_{n_k} - f_{n_j})(x) - (f_{n_k} - f_{n_j})(y)|}{|x-y|^{1/2}} \right)^{2\alpha} \\ &\times \left(\sup_{x,y\in[0,1]} (|f_{n_k} - f_{n_j}|(x) + |f_{n_k} - f_{n_j}|(y)) \right)^{1-2\alpha} \\ &\leq C||f_{n_k} - f_{n_j}||_{\sup}^{1-2\alpha} \end{aligned}$$

Therefore, f_{n_k} is Cauchy in the C^α norm, so since C^α is complete, f_{n_k} converges in C^α, as desired.

(3) It is not. For a counterexample, suppose f_n is a triangle with base of width $2n^{-2}$ centered at the origin and height n^{-1}. Then f_n' (distributionally) is $n(\chi_{[-n^{-2},0]} - \chi_{[0,n^{-2}]})$, which is bounded in L^2 (f itself goes to 0 in L^2). So f_n is a bounded sequence in H^1. On the other hand, if $m > n$, $||f_n - f_m||_{C^\alpha} \gtrsim 1$, so f_n cannot have a convergent sequence in C^α. *AN: I know this is vague - I'll try to fill it in with more detail later*

Exercise 80. *Extra 721 Problem:*

For $f \in L^2(\mathbb{R}^+)$, define $Tf(x) = \int_0^\infty \frac{f(y)}{x+2y}\,dy$. Prove that T is a bounded operator $L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$.

Solution 80. Changing variables, we see that it suffices to prove $\tilde{T}f(x) = \int_0^\infty \frac{f(y)}{x+y}\, dy$ is bounded $L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$. Applying the principle of duality, it suffices to prove that for any $f, g \in L^2(\mathbb{R}^+)$, $\int_0^\infty \int_0^\infty \frac{f(y)g(x)}{x+y}\, dx\, dy \le C||f||_{L^2}||g||_{L^2}$ for a fixed constant C. Write $I_k = [2^k, 2^{k+1}]$ for $k \in \mathbb{Z}$. Then we see that $\int_0^\infty \int_0^\infty \frac{f(y)g(x)}{x+y}\, dx\, dy = \sum_{k,j\in\mathbb{Z}} \int_{I_k}\int_{I_j} \frac{f(y)g(x)}{x+y}\, dx\, dy$. If $j \le k$ for $(x, y) \in I_k \times I_j$ $x + y \approx 2^k$, so $\int_{I_k}\int_{I_j} \frac{f(y)g(x)}{x+y}\, dx\, dy \approx 2^{-k}||f||_{L^1(I_k)}||g||_{L^1(I_j)}$. By Hölder's inequality, we see that $2^{-k}||f||_{L^1(I_k)}||g||_{L^1(I_j)} \le 2^{-k}2^{j/2}2^{k/2}||f||_{L^2(I_k)}||g||_{L^2(I_j)}$. Therefore, $\sum_{k\ge j}\int_{I_k}\int_{I_j} \frac{f(y)g(x)}{x+y}\, dx\, dy \approx \sum_{k\ge j} 2^{(j-k)/2}||f||_{L^2(I_k)}||g||_{L^2(I_j)}$.

Using Hölder's inequaltiy once more, we see that

$$\sum_{k\ge j} 2^{(j-k)/2}||f||_{L^2(I_k)}||g||_{L^2(I_j)} \le \left(\sum_{k\ge j} 2^{(j-k)}||f||^2_{L^2(I_k)}\right)^{1/2}\left(\sum_{k\ge j} 2^{(j-k)}||g||^2_{L^2(I_j)}\right)^{1/2}.$$

Because we are summing over the range $k, j \in \mathbb{Z}, k \le j$, fixing k and summing in j or fixing j and summing k, the 2^{j-k} always sums to 1. Therefore, we see that $\left(\sum_{k\ge j} 2^{(j-k)}||f||^2_{L^2(I_k)}\right)^{1/2} \le ||f||_{L^2(\mathbb{R}^+)}$ and $\left(\sum_{k\ge j} 2^{(j-k)}||g||^2_{L^2(I_j)}\right)^{1/2} \le ||g||_{L^2(\mathbb{R}^+)}$. Therefore,

$$\sum_{k\ge j} 2^{(j-k)/2}||f||_{L^2(I_k)}||g||_{L^2(I_j)} \le ||f||_{L^2(\mathbb{R}^+)}||g||_{L^2(\mathbb{R}^+)}.$$

We handle the $j \ge k$ sum similarly and arrive at the same conclusion, completing the problem.

Exercise 81. *Extra 721 Problem:*

Let $U = \{x \in \mathbb{R}^n : |x| < 1\}$ be the open unit ball in $\mathbb{R}^n$. Let $\rho : U \to \mathbb{R}$ be a smooth function such that $\rho(0) = 0, \nabla\rho(0) \ne 0$. Let $\Sigma = \{x \in U \mid \rho(x) = 0\}$. For $x \in U$, let $d(x) = \inf_{y\in\Sigma} |x - y|$.

(1) For $x \in V = \{x \in \mathbb{R}^n : |x| \le 1/2\}$, prove that there is a point $y \in \Sigma$ such that $d(x) = |x - y|$.
(2) For $x \in V \setminus \Sigma$ and for any $y \in \Sigma$ such that $d(x) = |x - y|$, prove that the vector $\nabla\rho(y)$ is a scalar multiple of $x - y$.
(3) Prove that there is an open set W with $0 \in W \subset V$ and a C^∞ function $\varphi : W \to \mathbb{R}$ such that for all $x \in W$, $|\varphi(x)| = d(x)$.

AN: This is a pretty old qual problem and part 2 and 3 feel more geometric (i.e. closer to a 761 problem) than most analysis qual problems now. Both require the implicit/inverse function theorem, but no theory beyond that.

Solution 81.

(1) Since $0 \in \Sigma$, $d(x) \le \frac{1}{2}$ for all $x \in V$, and if $d(x) = |x|$, then we can take $y = 0$. Now suppose $d(x) < |x|$. Choose η such that $2d(x) < \eta < 1$ and $y_n \in \Sigma$ such that $\lim_{n\to\infty} |x - y_n| = d(x)$. For n sufficiently large, $|x - y_n| - d(x) \le \eta - 2d(x)$, so $|x - y_n| \le \eta - d(x)$. It follows that for n sufficiently large, $y_n \in \overline{B_{\eta-d(x)}(x)} \cap \Sigma$. This is the intersection of a compact set and a closed set and hence is itself closed. Then we can find a subsequence $y_{n_k} \to y \in \Sigma$, and since $\lim_{k\to\infty} |x - y_{n_k}| = d(x)$, we know $|x - y| = d(x)$.

(2) We would like to prove that if $y \in \Sigma$ minimizes $|x-y|^2$, then $\nabla\rho(y)$ is a scalar multiple of $x-y$. Since y is a minimizer, we know that for any smooth path $\psi : (-\varepsilon, \varepsilon) \to \Sigma$ such that $\psi(0) = y$, $\frac{d}{dz}|x-\psi(z)|^2|_{z=0} = 0$. Since $|x-\psi(z)|^2 = x^2 + \psi(z)^2 - 2x \cdot \psi(z)$, we see that $\psi'(0) \cdot (x - \psi(0)) = \psi'(0) \cdot (x - y) = 0$. We will prove that for any v orthogonal to $\nabla\rho(y)$, we can find a smooth path ψ such that $\psi'(0) = v$. This would complete the problem, since if $v \cdot (x-y) = 0$ for any v orthogonal to $\nabla\rho(y)$, we must have that $x-y$ is colinear with $\nabla\rho(y)$.

For any curve $\psi : (-\varepsilon, \varepsilon) \to \Sigma$ such that $\psi(0) = y$, $\frac{d}{dz}\rho \circ \psi = 0$, since $\rho \circ \psi \equiv 0$. On the other hand, $\frac{d}{dz}\rho \circ \psi(0) = \nabla\rho(y) \cdot \psi'(0)$, so $\psi'(0)$ is orthogonal to $\nabla\rho(y)$. It suffices then to find $n-1$ functions $\psi_1, \ldots, \psi_n(0)$ such that $\psi_1'(0), \ldots, \psi_{n-1}'(0)$ are linearly independent.

For this, we will use the implict function theorem. We can assume $\nabla\rho(y) \neq 0$, since otherwise the $x-y$ is certainly colinear. Without loss of generality, assume $\frac{\partial}{\partial x_n}\rho(y) \neq 0$. Denote the first $n-1$ coordinates of y as y'. Then there is a ball $U = B(y', \varepsilon)$ and a smooth function $g : U \to \mathbb{R}$ so that $(z, g(z)) \in \Sigma$ for all $z \in U$. Define $\psi_i(\delta) = (\delta e_i + y', g(\delta e_i + y'))$ for $\delta \in (-\varepsilon, \varepsilon)$. Then $\psi_i'(0) = \left(e_i, \frac{\partial}{\partial e_i} g(y')\right)$. These are necessarily linearly independent if $e_i \neq e_j$, so as previously discussed, we are done.

(3) Use the implicit function theorem as described in the previous problem to find a neighborhood $B(0, \varepsilon)$ of 0 and a smooth function $g : B^{n-1}(0, \varepsilon) \to \mathbb{R}$ so that $(z, g(z)) \in \Sigma$ and $\nabla\rho(z, g(z)) \neq 0$. Now let $F : B^{n-1}(0, \varepsilon) \times \mathbb{R} \to V$ by $F(z, \alpha) = (z, g(z)) + \alpha \frac{\nabla\rho(z,g(z))}{|\nabla\rho(z,g(z))|}$. Using the inverse function theorem at $(0, 0)$ (I'll leave it to you to check the invertibility of $DF(0)$), we can find a neighborhood $W \subset B(0, \varepsilon)$ of 0 and a function $\Phi : W \to B^{n-1}(0, \varepsilon) \times \mathbb{R}$ such that $F \circ \Phi(y) = y$ for all $z \in W$. Then for $x \in W$, let $\varphi(x) = \Phi_n(x)$, where Φ_n denotes the n entry of Φ. Let's prove that $|\varphi(x)| = d(x)$. For each $x \in W$, we can find $y \in \Sigma$ such that $|x-y| = d(x)$. By the previous problem, it follows that there exists α such that $\alpha'\nabla\rho(y) = (x-y)$. Then $x = y + \alpha'\nabla\rho(y) = \alpha'|\nabla\rho(y)|\frac{\nabla\rho(y)}{|\nabla\rho(y)|}$. Denote $\alpha = \alpha'|\nabla\rho(y)|$. Since $y \in \Sigma \cap B(0, \varepsilon)$, $y = (z, g(z))$ for some $z \in B^{n-1}(0, \varepsilon)$. Then $x = F(z, \alpha)$. Then $\Phi(x) = (z, \alpha)$, so $\varphi(x) = \alpha$. But $d(x) = |x-y| = |\alpha'\nabla\rho(y)| = |\alpha| = |\varphi|(x)$, as desired.

Exercise 82. *Extra 721 Problem:*

Consider a differentiable function $f : \mathbb{R} \to \mathbb{R}$.

(1) Suppose the second derivative of f exists at x_0 (but not necessarily anywhere else). Show that $\lim_{h\to 0} \frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h^2} = f''(x_0)$.
(2) Suppose $\lim_{h\to 0} \frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h^2}$ exists. Recall that we have define f to be a differentiable function. Is it true that the second derivative of f exists at x_0?

Solution 82.

(1) We know that $f''(x_0) = \lim_{h\to 0} \frac{f'(x_0+h)-f'(x_0)}{h}$, so $f''(x_0) = \lim_{h\to 0} \frac{f'(x_0+h)-f'(x_0-h)}{2h}$. By L'Hopital's rule (which only requires differentiability and the numerator and denominator both going to zero), we know that

$$\lim_{h\to 0} \frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h^2} = \lim_{h\to 0} \frac{f'(x_0+h)-f'(x_0-h)}{2h}$$

which have already seen to be equal to $f''(x_0)$.

(2) No, consider, for example, $f(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$. This is differentiable every where with derivative $f'(x) = 2|x|$, so it is not twice differentiable at zero. On the other hand, $f(h) - f(-h) - 2f(0) = 0$ for all h, since f is odd, and therefore

$$\lim_{h \to 0} \frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2} = 0.$$

Exercise 83. *Extra 721 Problem:*

Show that there is no sequence $\{a_n\}_{n\in\mathbb{N}}$ of positive numbers such that $\sum_{n\in\mathbb{N}} a_n|c_n| < \infty$ if and only if c_n is bounded.

Hint: Suppose such a sequence exists and consider the map $T : \ell^\infty(\mathbb{N}) \to \ell^1(\mathbb{N})$ given by $[Tf]_n = a_n f(n)$. The set of f such that $f(n) = 0$ for all but finitely many n is dense in ℓ^1 but not in ℓ^∞.

Solution 83. Suppose such a sequence exists and define T as in the problem. Then by the uniform boundedness principle, $T : \ell^\infty(\mathbb{N}) \to \ell^1(\mathbb{N})$ is continuous. It also maps the set $A = \{\{a_n\}_{n=0}^\infty : a_n = 0 \text{ for all but finitely many } n\}$ bijectively to itself. Finally, it is surjective, since if $\{b_n\} \in \ell^1(\mathbb{N})$ but $\{b_n/a_n\}$ is unbounded, then $\sum_n b_n/a_n a_n$ cannot converge, and hence we must have that $\{b_n/a_n\}$ is bounded. Then it is an open mapping, by the open mapping theorem. Take an open set $O \subset \ell^\infty(\mathbb{N})$ not intersecting A (for example, $O = \{\{x_n\} : x_n > 1 \text{ for all } n\}$). Then $T(O)$ is open in $\ell^1(\mathbb{N})$, so it must intersect K, and hence there exists $x \in \ell^\infty(\mathbb{N}) \setminus K$ such that $T(x) \in K$, which is a contradiction, since $T(x)$ has the same number of zero entries as x.

Exercise 84. *Extra 721 Problem:*

Let $C([0, 1])$ denote the set of continuous functions on $[0, 1]$ equipped with the sup-norm. Prove that there exists a dense subset of $C([0, 1])$ consisting of functions are nowhere differentiable.

Solution 84. Define $A_n = \{f : \text{ there exists } x \in [0, 1] \text{ such that } |f(x) - f(y)| \leq n|x - y| \text{ for all } y \in [0, 1]\}$. Let's prove that A_n is closed and nowhere dense. Let f_m be a sequence in A_n converging to some f. Let x_m be a point such that $|f_m(x_m) - f_m(y)| \leq n|x_m - y|$ for all $y \in [0, 1]$. Since x_m is a bounded sequence, it has a convergent subsequence $x_{m_j} \to x$. Then $|f(x) - f(y)| = \lim_{j\to\infty} |f_{m_j}(x_{m_j}) - f_{m_j}(y)| \leq \lim_{j\to\infty} n|x_{m_j} - y| = n|x - y|$, so $f \in A_n$ and hence A_n is closed. Suppose $f \in A_n$. Fix $\varepsilon > 0$. There exists δ such that for all $|h| < \delta$, $|f(x + h) - f(x)| < \varepsilon$. Let $g(y)$ be a continuous function from $[0, 1]$ to $[-\varepsilon, \varepsilon]$ which is $-\varepsilon$ at x and ε at $x + c$, where c is sufficiently large that $c < \min(\delta, 1 - x, \varepsilon/(2n))$. Set $f_\varepsilon(y) = f(y) + g(y)$. Then $|f_\varepsilon - f| = |g| < \varepsilon$ and

$$|f_\varepsilon(x) - f_\varepsilon(x + c)| = |f(x) - f(x + c) + 2\varepsilon| \geq |2\varepsilon - \varepsilon| = \varepsilon.$$

Then $|f_\varepsilon(x) - f_\varepsilon(x + c)| \geq \varepsilon > nc$, so $f_\varepsilon \notin A_n$. It follows that A_n has empty interior. The intersection of A_n^c is an intersection of open dense sets, so it is residual and, in particular, non-empty. Suppose f in that intersection and f is differentiable at some point x with derivative α. Set $k = \lceil 2\alpha \rceil$. Then there exists some $\delta > 0$ such that for $y \in (x - \delta, x + \delta)$, $|f(y) - f(x)| \leq k|x - y|$. The function $y \mapsto \frac{|f(y)-f(x)|}{\delta}$ is continuous on the compact set $[0, 1] \setminus (x - \delta, x + \delta)$ and hence achieves some maximum $\leq m$. Set $n = \max(k, m)$. We know $|f(y) - f(x)| \leq k|x - y| \leq n|x - y|$. Since $\frac{|f(y)-f(x)|}{\delta} \leq m \leq n$ for $y \in [0, 1] \setminus (x -$

$\delta, x+\delta)$, $|f(y)-f(x)| \leq n\delta \leq n|x-y|$. Then $f \in A_n$, a contradiction. Thus, f is nowhere differentiable, as required.

Exercise 85. *Extra 721 Problem:*

Let H be a Hilbert space. For a linear space $Y \subset H$, define $Y^\perp = \{x \in H : (x,y) = 0\}$.

(1) Prove that if Y is closed, then $Y^\perp$ is a closed linear subspace of H.
(2) Prove that for any $x \in H$, a minimizing sequence for $\inf_{y\in Y} |x-y|$ is Cauchy. Conclude that we can uniquely write $x = x^{||} + x^\perp$ with $x^{||} \in Y$ and $x^\perp \in Y^\perp$.
(3) Prove that if $f : H \to \mathbb{R}$ is bounded and linear, then there exists $y \in H$ such that $f(x) = (x,y)$ for all x.

Solution 85.

(1) Take a convergent sequence $y_n \to y$ with $y_n \in Y^\perp$ for all n. Then $(y,x) = \lim_{n\to\infty}(y_n,x) = 0$ for any $x \in Y$. Hence, $Y^\perp$ is closed. For any $\alpha \in \mathbb{R}$ and $x,y \in Y^\perp$ and all $z \in Y$, $(\alpha x + y, z) = \alpha(x,z) + (y,z) = 0$. Therefore, $Y^\perp$ is a vector space.
(2) Let x_n be a minimizing sequence for $d = \inf_{x\in Y} ||x-y||_H$, that is $\lim_{n\to\infty} ||x_n - y||_H = \inf_{x\in Y} |x-y|$ and $x_n \in Y$ for all n (I switched the letter's around when I wrote the solution, sorry). Let's prove this is Cauchy. We have that $||x_n - x_m||_H^2 = ||x_n||^2 + ||x_m||^2 - 2\langle x_n, x_m\rangle$. We somehow have to use that $||x_n - y||_H^2$ is a minimizing sequence to make $2\langle x_n, x_m\rangle \to ||x_n||_H^2 + ||x_m||_H^2$. Since $||x_n - x_m||_H^2 \geq 0$, we know $||x_n||_H^2 + ||x_m||_H^2 \geq 2\langle x_n, x_m\rangle$, we will prove that in the limit, the opposite inequality holds as well. We can make this pop out by computing the distance between $x_n + x_m$ and y:

$$||\frac{x_n + x_m}{2} - y||_H^2 = \frac{||x_n||_H^2}{4} + \frac{||x_m||_H^2}{4} + \frac{\langle x_n, x_m\rangle}{2} + ||y||_H^2 - \langle x_n, y\rangle - \langle x_m, y\rangle.$$

On the other hand, for any $\varepsilon > 0$, we can choose N sufficiently large such that for any $n,m \geq N$, $||x_n - y||^2 \leq d^2 + \varepsilon \leq ||\frac{x_n+x_m}{2} - y||_H^2 + \varepsilon$. Then $\frac{||x_n-y||^2}{2} + \frac{||x_m-y||^2}{2} \leq ||\frac{x_n+x_m}{2} - y||_H^2 + \varepsilon$. Expanding both sides and simplifying, we see that $4\varepsilon + 2\langle x_n, x_m\rangle \geq ||x_n||^2 + ||x_m||^2$. The sending $\varepsilon \to 0$ and using the fact from before that $||x_n||_H^2 + ||x_m||_H^2 \geq 2\langle x_n, x_m\rangle$ for all n,m, we see that $||x_n - x_m||_H^2 \to 0$, and hence x_n is Cauchy. Denote it's limit by $x^{||}$.

Since each $x_n \in Y$ and Y is closed, $x^{||} \in Y$. Let $x^\perp = y - x^{||}$. Take $z \in Y$. We have that at $t = 0$, $\frac{d}{dt}||y - x^{||} + tz||_H^2 = 2\langle y, z\rangle - 2\langle x^{||}, z\rangle = 0$, since the minimum of $||y - x^{||} + tz||_H^2$ is at 0. It follows that $\langle y - x^{||}, z\rangle = 0$, so $\langle x^\perp, z\rangle = 0$. Hence, $x^\perp \in Y^\perp$.

To see that the decomposition is unique, suppose we can find a pair $x_0 \in Y$, $x_1 \in Y^\perp$ such that $x_0 + x_1 = y$. Then $x_0 - x^{||} = x_1 - x^\perp \in Y \cap Y^\perp$. But if $z \in Y \cap Y^\perp$, then $(z,z) = 0$, so $z = 0$ and hence, $x_0 = x^{||}, x_1 = x^\perp$. Thus, we have uniqueness.
(3) If $f \equiv 0$, then we can take $y = 0$. Otherwise, let $Y = \ker(f)$. Since $f \neq 0$, $Y \neq H$, so there exists $\tilde{y} \neq 0 \in Y^\perp$. It follows from the last problem that $Y \cap Y^\perp = \{0\}$, so for $x_1, x_2 \in Y^\perp$, $f(x_1), f(x_2) \neq 0$. Then we can find $\alpha \neq 0$ such that $\alpha f(x_1) + f(x_2) = f(\alpha x_1 + x_2) = 0$, and hence $\alpha x_1 + x_2 \in Y \cap Y^\perp$, so $\alpha x_1 + x_2 = 0$. Since $Y^\perp$ therefore cannot contain a pair of linearly independent vectors, we have $\dim(Y^\perp) = 1$. So take some element $y \in Y^\perp$. Then there exists $\alpha \neq 0$ such that

$f(y) = \alpha$, so $f(y) = \frac{\alpha}{||y||^2}(y, y)$. Set $\tilde{y} = \frac{\alpha y}{||y||^2}$, so that $f(y) = (y, \tilde{y})$. For all $x \in Y^\perp$, $x = \beta y$ for some $\beta \neq 0$, so

$$f(x) = f(\beta y) = \beta f(y) = \beta(y, \tilde{y}) = (\beta y, \tilde{y}) = (x, \tilde{y}).$$

For general $x \in H$, we use the previous problem to write $x = x^{||} + x^\perp$, with $x^\perp \in Y^\perp$ and $x^{||} \in Y$. Then $f(x^{||} + x^\perp) = f(x^\perp) = (x^\perp, \tilde{y}) = (x^\perp + x^{||}, \tilde{y})$, using the fact that $(y, x^{||}) = 0$ and $x^{||} \in \ker(f)$.

Exercise 86. *Extra 721 Problem:*

(1) Construct a set E such that on any interval non-empty finite interval I, $0 < |E \cap I| < |I|$.
(2) Prove or give a counterexample: there exists $\alpha \in (0, 1)$ and a measureable set E such that $\alpha|I| < |E \cap I| < |I|$ for every non-empty finite interval.

Solution 86.

(1) Enumerate a countable dense set in $\mathbb{R}$ (say, $\mathbb{Q}$) as $\{q_n : n \in \mathbb{N}\}$. We will construct our set E as follows: for each n, add $[q_n, q_n + \varepsilon_n]$ to the set and remove $[q_n - \varepsilon_n, q_n]$ from the set, where ε_n is a sequence satisfying $\varepsilon_n > 2\sum_{k>n} \varepsilon_k$. Then for any interval I, let I' be the middle third of I. Then there exists some m where $\varepsilon_m < |I'|$ and $q_m \in I'$. Then $E \cap I$ includes a positive measure subset of $[q_m, q_m + \varepsilon_m]$, since we only remove at most half the measure of that set in E after adding it to E, so $|E \cap I| > 0$. On the other hand, $E^c \cap I$ includes a positive measure subset of $[q_m - \varepsilon_m, q_m]$, since we only added half the measure of that set to E after removing it from E, so $|E \cap I| < |I|$.
(2) No. If there was such a set, then $f_r(x) = \frac{|E \cap [x-r.x+r]|}{2r} > \alpha$ for all x, r. Then $f(x) = \lim_{r\to 0} f_r(x) \geq \alpha$ for all x. By the Lebesgue Density theorem, it follows that $f(x) = 1$ almost everywhere, which implies that $|E \cap I| = |I|$ for all intervals I, a contradiction.

Exercise 87. *Extra 721 Problem:* Take a continuous function $K : [0, 1]^2 \to \mathbb{R}$ and suppose $g \in C([0, 1])$. Show that there exists a unique function $f \in C([0, 1])$ such that

$$f(x) = g(x) + \int_0^x f(y)K(x, y)\ dy.$$

Solution 87. First, suppose $K < 1$. Define the operator $T : C([0, 1]) \to C([0, 1])$ by $T(f)(x) = g(x) + \int_0^x f(y)K(x, y)\ dy$. Then $||T(f) - T(h)||_{\sup} \leq \int_0^1 |f - h|(y)|K|(x, y)\ dy < ||f - h||_{\sup}$, so T is a contraction. By the contraction mapping theorem, T has a fixed point f_0 satisfying the desired relation. Since T is a contraction, the solution must be unique.

Suppose $|K| < M$ for some large enough M. Then

$$||T(f) - T(h)||_{C[0,(2M)^{-1}]} \leq \int_0^{(2M)^{-1}} |f - h|(y)|K|(x, y)\ dy < ||f - h||_{C[0,(2M)^{-1}]}$$

Applying the Banach contraction principle from $[0, (2M)^{-1}]$, we find a fixed point f_0 on that interval. Now suppose we have defined f_0 on some interval $[0, a]$, so that for all $x \in [0, a]$, $f_0(x) = T(f_0)(x)$. Then we can extend f to $[0, a + (2M)^{-1}]$ as follows. Set $g_a(x) = g(x) + \int_0^a f_0(y)K(x, y)\ dy$ and $T_a : C([a, a + (2M)^{-1}]) \to C([a, a + (2M)^{-1}])$ by $T_a(f)(x) = g_a(x) + \int_a^{a+x} f(y)K(x, y)\ dy$. By the same reasoning as previously, we can find a fixed point f_a for T_a on $[a, a + (2M)^{-1}]$. Moreover, $f_a(a) = g_a(a) = g(a) + \int_0^a f_0(y)K(a, y)\ dy = f_0(a)$, so f_a does continuously extend f_0, so write the combined function as just f_0. Extending f_0

past a does not change the behavior of T up to a, so we still have $T(f_0)(x) = f_0(x)$ on $[0, a]$, while for $x \in [a, a + (2M)^{-1}]$, $T(f_0)(x) = g(x) + \int_0^a f_0(y)K(x, y)\ dy + \int_a^x f_0(y)K(x, y)\ dy = g(x) + \int_0^x f_0(y)K(x, y)\ dy$, as desired. After finitely many steps, we will have defined f on the entire interval $[0, 1]$.

Exercise 88. *Extra 721 Problem:* Let f be a continuous real-valued function on $\mathbb{R}$ satisfying $|f(x)| \leq \frac{1}{1+x^2}$. Define F on $\mathbb{R}$ by

$$F(x) = \sum_{n=-\infty}^{\infty} f(x + n)$$

(a) Prove that F is continuous and periodic with period 1.
(b) Prove that if G is continuous and periodic with period 1, then

$$\int_0^1 F(x)G(x)\ dx = \int_{-\infty}^{\infty} f(x)G(x)\ dx.$$

Solution 88.

(a) Define $F_N(x) = \sum_{n=-N}^{N} f(x + n)$. Then for any $k \in \mathbb{Z}$, $x \in [k, k + 2]$, and $N = N(k)$ sufficiently large $|F_N(x) - F(x)| \leq \sum_{|n|>N} \frac{1}{(x+n)^2+1} \lesssim \sum_{|n|>N} \frac{1}{n^2} \lesssim \frac{1}{N}$. It follows that F_N converges uniformly to F on $[k, k+2]$. Since each F_N is continuous on $[k, k+2]$, F is as well. Moreover, $|F(x) - F(x+1)| \leq |F_N(x) - F(x)| + |F_N(x+1) - F_N(x)| + |F_N(x+1) - F(x+1)|$. By what we've already shown, the first and third terms go to 0. The second term telescopes to $|f(x+N+1)-f(x-N)| \leq \frac{1}{(x+N+1)^2} + \frac{1}{(x-N)^2} \lesssim_x \frac{1}{N^2}$. It follows that $|F(x) - F(x + 1)| = 0$. Since x was arbitrary, F is periodic.

(b) First, note that since G is continuous and periodic, F is continuous and periodic, and f is absolutely integrable, both integrals always exist. Using the definitions in the previous part, as well as the periodicity of G, we see that

$$\begin{aligned}
\int_{-N}^{N+1} f(x)G(x)\ dx &= \sum_{n=-N}^{N} \int_n^{n+1} f(x)G(x)\ dx \\
&= \sum_{n=-N}^{N} \int_0^1 f(x + n)G(x)\ dx \\
&= \int_0^1 \sum_{n=-N}^{N} f(x + n)G(x)\ dx \\
&= \int_0^1 F_N(x)G(x)\ dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty} f(x)G(x)\ dx - \int_0^1 F(x)G(x)\ dx\right| \\
&\quad \leq \left|\int_{-\infty}^{\infty} f(x)G(x)\ dx - \int_{-N}^{N+1} f(x)G(x)\ dx\right| + \left|\int_0^1 F_N(x)G(x)\ dx - \int_0^1 F(x)G(x)\right|.
\end{aligned}$$

The first term goes to 0 by the integrability of $f(x)G(x)$, the second term is bounded about by $\sup_{x\in[0,1]} |F_N(x) - F(x)| \int_0^1 G(x)\ dx$, which goes to 0 by the first part.

Exercise 89. *Extra 721 Problem:* For $n \geq 2$ an integer, define $F(n)$ to be the function $F(n) = \max\{k \in \mathbb{Z} : 2^k/k \leq n\}$. Does $\sum_{n=2}^{\infty} 2^{-F(n)}$ converge?

Solution 89. We will write

$$\sum_{n=2}^{\infty} 2^{-F(n)} = \sum_{k=2}^{\infty} \left(\sum_{n:F(n)=k} 2^{-k} \right) = \sum_{k=2}^{\infty} \#\{n : F(n) = k\} 2^{-k}.$$

So we need to estimate $\#A_k$, where $A_k = \{n : F(n) = k\}$. For $n \in \mathbb{Z}$, $n \in A_k$ if and only if $\frac{2^k}{k} \leq n < \frac{2^{k+1}}{k+1}$. Then $\#A_k \approx \frac{2^{k+1}}{k+1} - \frac{2^k}{k}$, since the number of integers in an interval is within 2 of the length of the interval. It follows that $\sum_{k=2}^{\infty} \#\{n : F(n) = k\}2^{-k} \approx \sum_{k=2}^{\infty} \frac{2}{k+1} - \frac{1}{k}$. If this converged, then we could rearrange it to $\sum_{k=2}^{\infty} \frac{1}{k+1} - \frac{1}{k} + \sum_{k=2}^{\infty} \frac{1}{k+1}$. The former series is a telescoping sum converging to 1 and the latter diverges, but the sum of a convergent and a divergent sum cannot converge, so $\sum_{k=2}^{\infty} \frac{2}{k+1} - \frac{1}{k}$ cannot converge and hence neither can $\sum_{k=2}^{\infty} \#\{n : F(n) = k\}2^{-k}$, nor $\sum_{n=2}^{\infty} 2^{-F(n)}$.

Exercise 90. *Extra 721 Problem:* For a_n, b_n sequence in $\ell^2(\mathbb{N})$, prove that

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_n b_k}{n+k} \leq C||a||_2||b||_2.$$

Solution 90. For $m \in \mathbb{N}$, let $I_m = \{n \in \mathbb{N} : 2^{m-1} \leq n \leq 2^m\}$. Write

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_n b_k}{n+k} = \sum_{\substack{m_1,m_2 \in \mathbb{N}\times\mathbb{N} \\ m_1 \geq m_2}} \sum_{(n,k)\in I_{m_1}\times I_{m_2}} \frac{a_n b_k}{n+k} + \sum_{\substack{m_1,m_2 \in \mathbb{N}\times\mathbb{N} \\ m_1 < m_2}} \sum_{(n,k)\in I_{m_1}\times I_{m_2}} \frac{a_n b_k}{n+k}.$$

We will prove $\sum_{\substack{m_1,m_2 \in \mathbb{N}\times\mathbb{N} \\ m_1 \geq m_2}} \sum_{(n,k)\in I_{m_1}\times I_{m_2}} \frac{a_n b_k}{n+k} \leq C||a_n||_{\ell^2}||b_n||_{\ell^2}$, the other sum will follow similarly. If $m_1 \geq m_2$, for $(n,k) \in I_{m_1} \times I_{m_2}$, $\frac{1}{n+k} \approx 2^{-m_1}$. It follows by Fubini's theorem $\sum_{(n,k)\in I_{m_1}\times I_{m_2}} \frac{a_n b_k}{n+k} \leq 2^{-m_1}||a_n||_{\ell^1(I_{m_1})}||b_n||_{\ell^1(I_{m_2})}$. By Hölder's inequality and the fact that $\#I_m = 2^m$, we have

$$2^{-m_1}||a_n||_{\ell^1(I_{m_1})}||b_n||_{\ell^1(I_{m_2})} \leq 2^{-(m_1-m_2)/2}||a_n||_{\ell^2(I_{m_1})}||b_n||_{\ell^2(I_{m_2})}.$$

Using Hölder's inequality and Fubini-Tonelli, we see that

$$\begin{aligned}
\sum_{\substack{m_1,m_2 \in \mathbb{N}\times\mathbb{N} \\ m_1 \geq m_2}} \sum_{(n,k)\in I_{m_1}\times I_{m_2}} \frac{a_n b_k}{n+k} &\leq \sum_{m_1 \geq m_2} 2^{-(m_1-m_2)/2}||a_n||_{\ell^2(I_{m_1})}||b_n||_{\ell^2(I_{m_2})} \\
&\leq \left(\sum_{m_1 \geq m_2} 2^{-(m_1-m_2)/2}||a_n||^2_{\ell^2(I_{m_1})} \right)^{1/2} \left(\sum_{m_1 \geq m_2} 2^{-(m_1-m_2)/2}||b_n||^2_{\ell^2(I_{m_2})} \right)^{1/2} \\
&\leq C||a_n||_{\ell^2(\mathbb{N})}||b_n||_{\ell^2(\mathbb{N})}
\end{aligned}$$

To see the final inequality, we sum first in the variable which is only in the power of 2. No matter than value of the other variable, the sum is always bounded above by $c = \sum_{n=1}^{\infty} \sqrt{2}^{-n} < \infty$. What is left is then bounded above by $c^{1/2} \left(\sum_{m_1 \in \mathbb{N}} ||a_n||^2_{\ell^2(I_{m_1})} \right)^{1/2} = c^{1/2}||a_n||_{\ell^2(\mathbb{N})}$, as desired. This completes the problem.

Exercise 91. *Extra 721 Problem:* Let x_n be a sequence in a Hilbert space H. Suppose that x_n converges weakly to x as $N \to \infty$. Prove that there is a subsequence x_{n_k} such that

$$N^{-1} \sum_{k=1}^{N} x_{n_k}$$

converges in norm to x

Solution 91. Recall that if x_n converges to x weakly and $||x_n|| \to ||x||$, then $x_n \to x$, since $||x_n - x|| = \langle x_n - x, x_n - x\rangle = ||x_n||^2 + ||x||^2 - 2\langle x_n, x\rangle$. For any subsequence x_{n_k}, by the standard Césaro sum proof,

$$\left\langle \frac{1}{N} \sum_{k=1}^{N} x_{n_k}, h \right\rangle = \frac{1}{N} \sum_{k=1}^{N} \langle x_{n_k}, h\rangle \to \langle x, h\rangle$$

for all h, so it suffices to find a subsequence x_{n_k} so that $\left|\left| \frac{1}{N} \sum_{k=1}^{N} x_{n_k} \right|\right|^2 \to ||x||^2$. We can expand $\left|\left| \frac{1}{N} \sum_{k=1}^{N} x_{n_k} \right|\right|^2$ to $\frac{1}{N} \sum_{k=1}^{N} \left(\frac{1}{N} \sum_{j=1}^{N} \langle x_{n_k}, x_{n_j}\rangle \right)$. We can find a subsequence x_{n_k} so that for $j \geq \sqrt{k}$, $\langle x_{n_k}, x_{n_j}\rangle - \langle x_{n_k}, x\rangle \leq \frac{1}{k}$. Note that since x_n converges weakly, $||x_n||^2$ is bounded by some b, and hence $\langle x_n, x_m\rangle \leq b^2$ for all n, m. Then

$$\frac{1}{N} \sum_{k=1}^{N} \left(\frac{1}{N} \sum_{j=1}^{N} \langle x_{n_k}, x_{n_j}\rangle \right) - ||x||^2 = \frac{1}{N} \sum_{k=1}^{N} \left(\frac{1}{N} \sum_{j=1}^{N} \langle x_{n_k}, x_{n_j}\rangle - \langle x_{n_k}, x\rangle \right) + \frac{1}{N} \sum_{k=1}^{N} (\langle x_{n_k}, x\rangle - \langle x, x\rangle)$$

The latter term decays to 0 since it is the Cesaro sum of a sequence that converges to 0. For the first term,

$$\left| \sum_{j=1}^{N} \langle x_{n_k}, x_{n_j}\rangle - \langle x_{n_k}, x\rangle \right| \leq \left| \sum_{j=1}^{\sqrt{k}} \langle x_{n_k}, x_{n_j}\rangle - \langle x_{n_k}, x\rangle \right| + \left| \sum_{j=\sqrt{k}+1}^{N} \langle x_{n_k}, x_{n_j}\rangle - \langle x_{n_k}, x\rangle \right| \leq \sqrt{k} b^2 + \frac{N}{k}.$$

Then

$$\left| \frac{1}{N} \sum_{k=1}^{N} \left(\frac{1}{N} \sum_{j=1}^{N} \langle x_{n_k}, x_{n_j}\rangle - \langle x_{n_k}, x\rangle \right) \right| \leq \frac{1}{N} \sum_{k=1}^{N} \frac{\sqrt{k} b^2}{N} + \frac{1}{k}.$$

As $N \to \infty$, the final sum goes to 0, completing the proof.