ALL PROBLEMS

Exercise 1. Let $K : [a,b] \times [a,b] \to \mathbb{R}$ be a differentiable function such that

$$\max_{[a,b]^2} |K(x,t)| \le 1, \max_{[a,b]^2} \left| \frac{\partial K}{\partial x}(x,t) \right| \le 1.$$

Consider the space $C[a,b]$ of continuous functions on $[a,b]$ with the sup-norm. For $f \in C[a,b]$, define

$$Af(x) = \int_a^b K(x,t) f(t) \, dt.$$

(1) Prove that $\{Af : \max_{[a,b]} |f(x)| \le 1\}$ is a totally bounded subset of $C[a,b]$.
(2) If in (1) we drop the assumption $\max_{[a,b]^2} \left| \frac{\partial K}{\partial x}(x,t) \right| \le 1$ and keep the other assumptions, does $\{Af : \max_{[a,b]} |f(x)| \le 1\}$ have to be a totally bounded subset of $C[a,b]$?

Exercise 2. For a sequence (a_k) let $s_n = \sum_{k=1}^n a_k$ and $\sigma_L = \frac{1}{L} \sum_{n=1}^L s_n$. We say that $\sum_{k=1}^\infty a_k$ is Cesáro summable to S if $\lim_{L\to\infty} \sigma_L = S$.

(1) Prove: $s_n - \sigma_n = \frac{(n-1)a_n + (n-2)a_{n-1} + \cdots + a_2}{n}$.
(2) Prove: If $\sum_{k=1}^\infty a_k$ is Cesáro summable to S and if $\lim_{k\to\infty} k a_k = 0$, then $\sum_{k=1}^\infty a_k$ converges and $\sum_{k=1}^\infty a_k = S$.

Exercise 3. Consider the space $C([0,10])$ of continuous functions on $[0,10]$, and for a given large number L consider the metric $d_L(f,g) = \max_{x\in[0,10]} e^{-Lx} |f(x) - g(x)|$.

(1) Argue that $C([0,10])$ with the metric d_L is a complete metric space.
(2) Show that there is a unique function which is continuous on $[0,10]$ and satisfies

$$f(x) = -15 + \cos(x) \int_0^x e^{e^{tx}} f(t) \, dt$$

for all $x \in [0,10]$.

Exercise 4. Let

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < x_1^2 \le 1/2\}.$$

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x) = (x_1^2 + x_2^2)^{-b/2} |\log(x_1^2 + x_2^2)|^{-\gamma}.$$

Determine for which values $b > 0, \gamma \in \mathbb{R}$, $\int_\Omega f(x) \, dx$ is finite.

Exercise 5. Suppose $f_n \in L^1(\mathbb{R})$ and the sequence ε_n of positive real numers satisfies $\lim_{n\to\infty} \varepsilon_n = 0$. Define the sets $E_n = \{x : |f_n(x)| \ge \varepsilon_n\}$ and assume that $\sum_n m(E_n) < \infty$. Prove that

(1) $\lim_{n\to\infty} f_n = 0$ Lebesgue a.e. on $\mathbb{R}$.
(2) For every $\delta > 0$, there is a set Ω such that $m(\Omega) < \delta$ and $\lim_{n\to\infty} f_n = 0$ uniformly on $\mathbb{R} \setminus \Omega$.

Exercise 6. Let $E \subset \mathbb{R}$. Suppose $g, f_n \in L^1(E)$, $\sup_n ||f_n||_{L^1} < \infty$ and $\lim_{n\to\infty} f_n = 0$ in Lebesgue measure. Prove that

$$\lim_{n\to\infty} \int_E \sqrt{|f_n g|}\ dx = 0.$$

Exercise 7. Suppose that on a set E of finite measure, $f_n \to f$ in measure and $g_n \to g$ in measure and f is finite a.e.. Prove that $f_n g_n \to fg$ in measure on E.

Hint: Can you prove $f_n^2 \to f^2$ in measure?

Exercise 8. Let $X = \{P : \mathbb{R} \to \mathbb{R} | P \text{ is a polynomial}\}$. Prove that there does not exist a norm $|| \cdot ||$ on X such that $(X, || \cdot ||)$ is a Banach space.

Exercise 9. Suppose $g_n \in S(\mathbb{R}^2)$ and $\lim_{n\to\infty} ||g_n||_{L^2}(\mathbb{R}^2) = 0$. Show that there are $f_n \in C^2(\mathbb{R}^2)$ such that $\Delta f_n = f_n + g_n$ and f_n satisfies

(1) $\lim_{n\to\infty} f_n(0,0) = 0$.
(2) $\lim_{n\to\infty} ||\partial^2_{x_1 x_2}(f_n)||_{L^2(\mathbb{R}^2)} = 0$.

Exercise 10. The following distributions u, v on $\mathbb{R}^2$ are defined by pairing with Schwartz functions via

$$\langle u, \phi \rangle = \int_0^2 \phi(0,t)\ dt$$

$$\langle v, \phi \rangle = \int_0^2 \phi(t,0)\ dt$$

Show that the convolution $u * v$ can be identified with a finite, absolutely continuous measure μ. Find $g \in L^1(\mathbb{R}^2)$ such that $\int \phi\ d\mu = \int \phi g\ dx$ for all Schwartz functions ϕ.

Exercise 11. Suppose that E is a measurable set of real numbers with arbitrarily small periods (that is, there exists a sequence of real numbers $p_i \to 0$ such that $E + p_i = E$). Prove that either E or it's complement has measure 0.

Exercise 12. Let $\mathbf{T} \subset \mathbb{C}$ be the unit circle. We say that $G \subset \mathbf{T}$ is a subgroup of $\mathbf{T}$ if $1 \in G$, $\zeta_1, \zeta_2 \in G$ implies $\zeta_1\zeta_2 \in G$ and $\zeta_1 \in G$ implies $\zeta_1^{-1} \in G$.

(1) What are the compact subgroups of $\mathbf{T}$?
(2) Give an example of an infinite subgroup $G \subsetneq \mathbf{T}$.
(3) Prove or give a counterexample: there are no measurable subgroups $G \subsetneq \mathbf{T}$ with $|G| > 0$.

Exercise 13. Let $\{a_n\}$ be a convergent sequence of complex numbers and let $\lim_{n\to\infty} a_n = L$. Let

$$c_n := \frac{1}{n^5} \sum_{k=1}^{n} k^4 a_k.$$

Prove that c_n converges and determine its limit.

Exercise 14. Let $K : [a,b]\times[a,b] \to \mathbb{R}$ be a continuous function. Consider the space $C[a,b]$ of continuous functions on $[a,b]$ with the sup-norm. For $f \in C[a,b]$, define

$$S_K f(x) = \int_a^b K(x,t)f(t)\ dt.$$

(1) Is $\{S_K f : \max_{[a,b]} |f(x)| \le 1\}$ necessarily a totally bounded subset of $C[a,b]$?
(2) Let f_n be a sequence of continuous functions on $[a,b]$ satisfying

$$\sup_n \sup_{x\in[a,b]} |f_n(x)| \le 1.$$

Does the sequence $S_K f_n$ necessarily have a convergent subsequence? Give a proof or counterexample.
(3) Let K_n be a sequence of continuous functions on $[a,b]\times[a,b]$ and assume that

$$\sup_n \max\{|K_n(x,y)| : (x,y) \in [a,b]\times[a,b]\} \le 1.$$

Let $f \in C[a,b]$. Does the sequence $S_{K_n} f$ necessarily have a convergent subsequence in $C[a,b]$? Prove or give a counterexample.

Exercise 15. Show that for $\alpha \neq 0$,

$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty} \frac{\alpha}{\alpha^2+n^2} = \frac{e^{2\pi\alpha}+1}{e^{2\pi\alpha}-1}.$$

Hint: Apply the Poisson summation formula to the function $f(x) = e^{-c|x|}$, for an appropriate choice of c.

Exercise 16. Prove that there are two functions $f_1, f_2 \in C[0,1]$ that solve the following system of equations for all $x \in [0,1]$,

$$20f_1(x) + 3f_2(x) = \sin(x) + \int_0^1 \sin(xt)\sin(f_1(t))\ dt$$

$$-f_1(x) + 10f_2(x) = \cos(x) - \int_0^{1/2} \cos(xt)\cos(f_2(t))\ dt.$$

Exercise 17. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set with finite Lebesgue measure $m(E)$. Define $f(r) = m((E+r)\cap E)$. Prove that f is continuous.

Exercise 18. Take $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^n$ and let $X+Y = \{x+y : x\in X, y \in Y\}$.

(1) Assume X is closed and Y is compact. Prove that $X+Y$ is closed.
(2) If Y is closed but not compact, is $X+Y$ closed? Prove or give a counterexample.

Exercise 19. For $f \in L^2$, let $F(x) = \int_0^x f(t)\ dt$.

(1) Prove that

$$\int_0^1 \left(\frac{F(x)}{x}\right)^2\ dx \le 4\int_0^1 f^2(x)\ dx.$$

(2) For $x \in [0,1]$
$$Af(x) = \frac{1}{x\sqrt{1+|\log(x)|}} \int_0^x f(t)\ dt.$$
Prove that if f_n is a sequence of continuous functions on $[0,1]$ with $\sup_n ||f_n||_{L^2([0,1])} \leq 1$, then Af_n has a subsequence converging in the $L^2([0,1])$ norm.

Exercise 20. Prove that there does not exist $f \in L^1(\mathbb{R})$ such that $g * f(x) = g(x)$ for any $g \in C_0(\mathbb{R})$ and $x \in \mathbb{R}$.

Exercise 21. Let $I = [0,1)$ and given $N \in \mathbb{N}$ consider the dyadic intervals $I_{j,N} = [j2^{-N}, (j+1)2^{-N})$ for $j \in \{0, 1, \dots, 2^N - 1\}$. For a function $f \in L^1(I)$, define a sequence of function $E_N f : I \to \mathbb{R}$ by
$$E_N f(x) = 2^N \int_{I_{j,N}} f(t)\ dt \quad \text{ for } x \in I_{j,N}.$$
Show that $\lim_{N\to\infty} E_N f(x) = f(x)$ for a.e. $x \in I$.

Exercise 22. Suppose f, f_n are Lebesgue measureable functions on $[0,1]$ finite a.e.. Show that $f_n \to f$ in measure if and only if
$$\lim_{n\to\infty} \int_0^1 \frac{|f_n - f|(x)}{1 + |f_n - f|(x)}\ dx = 0.$$

Exercise 23. For $f, g \in L^2[0,1]$, let $\langle f, g\rangle = \int_0^1 f(x)\overline{g}(x)\ dx$ and set
$$g_n(x) = \frac{n^{2/3}\sin(n/x)}{xn+1}.$$
Does there exists $\alpha > 0$ such that
$$\sum_{n=1}^{\infty} |\langle f, g_n\rangle|^\alpha < \infty.$$
hold for every $f \in L^2$? *Hint: is $||g_n||_{L^2}$ a bounded sequence?*

Exercise 24. Prove that there is a distribution $u \in \mathcal{D}'(\mathbb{R})$ so that its restriction to $(0,\infty)$ is given by
$$\langle u, f\rangle = \int_0^\infty x^{-2}\cos(x^{-2}) f(x)\ dx$$
for all $f \in C^\infty(\mathbb{R})$ compactly supported on $(0,\infty)$ and $\langle u, f\rangle = 0$ for all $f \in C^\infty(\mathbb{R})$ compactly supported on $(-\infty, 0)$.

Exercise 25. Consider the series
$$\sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{x}{n}\right)$$
.

(1) Does it converge uniformly on $[0,1]$?
(2) Does it converge uniformly on $[0,\infty)$?

Exercise 26. Can one find a bounded sequence of real numbers $x_n, n \in \mathbb{Z}$ that satisfies
$$x_n = \sin(n) + 0.5x_{n-1} + 0.4\sin(x_{n+1})$$
for every $n \in \mathbb{Z}$?

Exercise 27. Suppose S is the set of real-valued functions continuous g on $[0,1]$ that satisfy two conditions:

$$\left|\int_0^1 g(x)\ dx\right| \le 1$$

and

$$|g(x) - g(y)| \le |x-y|^{1/2}$$

for each $x, y \in [0,1]$. Consider the functional

$$F(g) = \int_0^1 (1-5x^2)g^{10}(x)\ dx.$$

Is F bounded on S? Does it acheive it's maximum on S?

Exercise 28. Let $f_n : X \to \overline{\mathbb{R}}$ be a sequence of measurable functions on a finite measure space X, so that $|f_n(x)| < \infty$ for almost every $x \in X$. Show that there is a sequence A_n of positive real numbers so that

$$\lim_{n\to\infty} \frac{f_n(x)}{A_n} = 0$$

almost everywhere. *Hint: Borel-Cantelli.*

Exercise 29. Let $I = [a,b]$ and let $L^2(I)$ be the space of square-integrable functions on $[a,b]$ with scalar product $\langle f, g\rangle = \int_a^b f(t)\overline{g}(t)\ dt$. Let $p_0, p_1, \ldots$ be a sequence of real-valued polynomials p_n of degree exactly n such that

$$\int_a^b p_j(x)p_k(x)\ dx = \begin{cases} 0 & j \ne k \\ 1 & j = k \end{cases}.$$

Prove that $\{p_n\}_{n=0}^{\infty}$ is a *complete* orthonormal system.

Exercise 30. For $x, y \in \mathbb{R}$, let $K(y) = \pi^{-1}(1+y^2)^{-1}$, and for $t > 0$ let

$$P_t f(x) = \int_{-\infty}^{\infty} t^{-1}K(t^{-1}y)f(x-y)\ dy.$$

(1) Show that if f is continuous and compactly supported, then

$$\lim_{t\to 0^+} \sup_{x\in\mathbb{R}} |P_t f(x) - f(x)| = 0.$$

(2) Let $p \ge 1$. For $f \in L^p(\mathbb{R})$ denote by Mf the Hardy-Littlewood maximal function of f. Show that there is a constant $C > 0$ so that for all $f \in L^p(\mathbb{R})$ the inequality

$$|P_t f(x)| \le CMf(x)$$

holds for every $x \in \mathbb{R}$ and every $t > 0$.

(3) If $f \in L^1(\mathbb{R})$, prove that $\lim_{t\to 0+} P_t f(x) = f(x)$ for almost every $x \in \mathbb{R}$.

Exercise 31. On $\mathbb{R} \setminus \{0\}$ define $f(x) = |x|^{-7/2}$. Find a tempered distribution $h \in \mathcal{S}'(\mathbb{R})$ so that $f = h$ on $\mathbb{R} \setminus \{0\}$.

Exercise 32. Let X, Y be σ-finite measure space with positive measures $d\mu, d\nu$ respectively and let K be a measurable function on $X \times Y$. Let u and w be nonnegative measurable functions on X, Y respectively. Assume that $u(x) > 0$ and $w(y) > 0$ a.e.. Suppose $1 < p < \infty$

$$\int_X |K(x,y)| u(x) d\mu(x) \leq Bw(y), \quad \nu - \text{a.e.}$$

$$\int_Y |K(x,y)| w(y)^{1/(p-1)} d\nu(y) \leq Au(x)^{1/(p-1)} \quad \mu - \text{a.e.}$$

Let T be defined by $Tf(x) = \int_Y K(x,y) f(y)\, d\nu(y)$. Show that T maps $L^p(Y)$ to $L^p(X)$ with operator norm bounded by $A^{1-1/p} B^{1/p}$.

Exercise 33. Let f_n be a sequence of continuous functions on $I = [0,1]$. Suppose that for every $x \in I$ there exists an $M(x) < \infty$ so that $|f_n(x)| \leq M(x)$ for all $n \in \mathbb{N}$. Show then that $\{f_n\}$ is uniformly bounded on some interval, that is there exists $M \in \mathbb{R}$ and an interval $(a,b) \subset I$ so that $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and $x \in (a,b)$.

Exercise 34. Suppose that $f_n : [0,1] \to \mathbb{R}$ is a sequence of continuous functions each of which has continuous first and second derivatives on $(0,1)$. Prove: If

$$f(x) = \lim_{n \to \infty} f_n(x) \quad \text{for all} \quad x \in [0,1]$$

and

$$\sup_{n \geq 1} \max_{0<x<1} |f_n''(x)| < \infty,$$

then f' exists and is continuous on $(0,1)$.

Exercise 35. A function $f : U \to \mathbb{R}$ defined on a subset $U \subset \mathbb{R}^n$ is

- *locally bounded* if for all $x \in U$ there exists $\varepsilon, R > 0$ such that $|f(y)| \leq R$ for all $y \in U$ with $|x - y| < \varepsilon$,
- *globally bounded* if there exists $R > 0$ such that $|f(y)| \leq R$ for all $y \in U$.

Prove: If $U \subset \mathbb{R}^n$, then the following are equivalent:

(1) U is compact,
(2) every locally bounded function $f : U \to \mathbb{R}$ is globally bounded.

Exercise 36. Let $\mathcal{K}$ denote the collection of compact subsets of $[0,1]$. Define the Hausdorff metric on $\mathcal{K}$ by

$$d(K_1, K_2) = \sup_{x \in K_1} \inf_{y \in K_2} |x - y| + \sup_{x \in K_2} \inf_{y \in K_1} |x - y|.$$

Prove that $(\mathcal{K}, d)$ is a complete metric space.

Exercise 37. Prove that any open set $U \subset \mathbb{R}^n$ can be expressed as a countable union of rectangles.

Exercise 38. Suppose that $f \in L^1(\mathbb{R})$. Consider the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = \int_{\mathbb{R}} \frac{f(y)}{1 + |xy|}\, dy.$$

(1) Prove that F is continuous,
(2) Prove that if $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, then $F \in L^p(\mathbb{R})$ for all $p > 2$.

Exercise 39. For any $n \geq 1$, show that there exists closed sets $A, B \subset \mathbb{R}^n$ with $|A| = |B| = 0$, but $|A+B| > 0$ (as usual $A + B = \{a + b : a \in A, b \in B\}$).

Exercise 40. Suppose $E \subset \mathbb{R}^d$ is a given set and O_n is the open set $O_n = \{x : d(x, E) < 1/n\}$.

(1) Show that if E is compact, then $|E| = \lim_{n\to\infty} |O_n|$.
(2) Is the statement false for E closed and unbounded?
(3) Is the statement false for E open and bounded?

Exercise 41. Let $\mathcal{D}'(\mathbb{R})$ denote the space of distributions on $\mathbb{R}$ with the weak-$*$ topology. Determine the limit in $\mathcal{D}'(\mathbb{R})$ of the sequence of functions in $\mathbb{R}$:

$$\lim_{n\to\infty} \sqrt{n} e^{\frac{i}{2}nx^2}.$$

Exercise 42. For $s > \frac{1}{2}$, let $H^s(\mathbb{R}^n)$ denote the Sobolev space

$$H^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, d\mu(\xi) < \infty\}$$

(where μ is the Lebesgue measure and $\hat{f}$ is the Fourier transform of f). Show that if $u, v \in H^s(\mathbb{R}^n)$ for $s > n/2$, the $uv \in H^s(\mathbb{R}^n)$ and

$$||uv||_{H^s(\mathbb{R}^n)} \leq C||u||_{H^s(\mathbb{R}^n)}||v||_{H^s(\mathbb{R}^n)}$$

for a constant C depending only on s and n.

Exercise 43. Let $x_1, \ldots, x_{n+1}$ be pairwise distinct real numbers. Prove that there exists $C > 0$ such that: if $P : \mathbb{R} \to \mathbb{R}$ is a polynomial with degree at most n, then

$$\max_{x\in[0,1]} |P(x)| \leq C \max\{|P(x_1)|, \ldots, |P(x_{n+1})|\}.$$

Exercise 44. Given a real number x, let $\{x\}$ denote the fractional part of x. Suppose α is an irrational number and define $T : [0, 1] \to [0, 1]$ by

$$T(x) = \{x + \alpha\}.$$

Prove: If $A \subset [0, 1]$ is measurable and $T(A) = A$, then $|A| \in \{0, 1\}$.

Exercise 45. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable, real-valued functions on a measure space X such that $f_n \to f$ pointwise as $n \to \infty$, where $f : X \to \mathbb{R}$, and suppose that for some constant $M > 0$,

$$\int |f_n| \, d\mu \leq M \quad \text{for all } n \in \mathbb{N}.$$

(1) Prove that
$$\int |f| \, d\mu \leq M.$$
(2) Give an example to show that we may have $\int |f_n| \, d\mu = M$ for every $n \in \mathbb{N}$, but $\int |f| \, d\mu < M$.
(3) Prove that
$$\lim_{n\to\infty} \int ||f_n| - |f| - |f_n - f|| \, d\mu = 0.$$

Exercise 46. Consider the following equation for an unknown function $f : [0,1] \to \mathbb{R}$:

$$f(x) = g(x) + \lambda \int_0^1 (x-y)^2 f(y)\ dy + \frac{1}{2}\sin(f(x)).$$

Prove that there exists a number λ_0 such that for all $\lambda \in [0, \lambda_0)$ and all continuous functions g on $[0,1]$, the equation has a continuous solution.

Exercise 47. Given $\alpha \geq 0$ the α-dimensional Hausdorff measure of a set $X \subset \mathbb{R}^n$ is

$$\mathcal{H}^\alpha(X) = \liminf_{r\to 0}\{\sum_{i=1}^\infty r_i^\alpha : X \subset \bigcup_{i=1}^\infty B(x_i, r_i), r_i < r \text{ for all } i\}$$

and the Hausdorff dimension is $\dim_H(X) = \inf\{\alpha \geq 0 : \mathcal{H}^\alpha(X) = 0\}$.

Prove the following:

(1) If $X \subset \mathbb{R}^n$ and μ is a finite Borel measure on $\mathbb{R}^n$ such that $\mu(X) > 0$ and $\mu(B(x,r)) \leq r^\alpha$ for all open balls $B(x,r)$, then $\dim_H(X) \geq \alpha$.
(2) If $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is the unit circle, the $\dim_H(S^1) = 1$.

Exercise 48. Let $X = [0,1]$ with Lebesgue measure and $Y = [0,1]$ with counting measure. Give an example of a measurable function $f : X \times Y \to [0, \infty)$ for which Fubini's theorem does not apply. (This example shows that the theorem is not valid if the hypothesis of σ-finiteness is omitted.)

Exercise 49. For $s > \frac{1}{2}$ let $H^s(\mathbb{R}^n)$ denote the Sobolev space

$$H^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2\ d\mu(\xi) < +\infty\}$$

(where μ is the Lebesgue measure and $\hat{f}$ is the Fourier transform of f). Use the Fourier transform to prove that if $u \in H^s(\mathbb{R}^n)$ for $s > n/2$, then $u \in L^\infty(\mathbb{R}^n)$, wih the bound

$$||u||_{L^\infty} \leq C||u||_{H^s(\mathbb{R}^n)}$$

for a constant C depending only on s and n.

Exercise 50. Assume that X is a compact metric space and $T : X \to X$ is a continuous map. Let $\mathcal{M}_1(T)$ denote the set of Borel probability measures on X such that $T_*\mu = \mu$. Prove:

(1) $\mathcal{M}_1(T) \neq \emptyset$.
(2) If $\mathcal{M}_1(T) = \{\mu\}$ consists of a single measure μ, then

$$\int_X f\ d\mu = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n(x)$$

for every continuous function $f : X \to \mathbb{R}$ and point $x \in X$.

Exercise 51. Find the Fourier transform of the following function: $f \in \mathbb{R}^2$:

$$f(x) = e^{ix\xi_0}|x - x_0|^{-1}.$$

Exercise 52. Let $f \in C^1([0,1])$. Show that for every $\varepsilon > 0$ there exists a polynomial p such that

$$||f - p||_\infty + ||f' - p'||_\infty < \varepsilon.$$

Exercise 53. Let $a = (a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers. Prove that the set

$$X = \{(x_n)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N}) : x_n \in [0, a_n] \text{ for all } n \in \mathbb{N}\}$$

is compact in the $\ell^1(\mathbb{N})$ norm if and only if $(a_n)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N})$.

Exercise 54. Let $z_1, z_2, \dots, z_n$ be points on the unit circle $\mathbf{T} = \{|z| = 1\}$ in the complex plane. Let $E \subset \mathbf{T}$ satisfy $m(E) > 2\pi(1 - \frac{1}{n})$. Prove that E can be rotated so that all the points z_k fall into the rotated set, i.e., that there exists $\alpha \in \mathbf{T}$ such that $\alpha z_k \in E$ for $k = 1, 2, \dots, n$.

Exercise 55. Let $f \in L^1(\mathbb{R})$ satisfy $\int_a^b f(x)\, dx = 0$ for any two rational numbers $a < b$. Does it follow that $f(x) = 0$ for almost every x?

Exercise 56. Let σ be a Borel probability measure on $[0, 1]$ satisfying

(1) $\sigma([1/3, 2/3]) = 0$;
(2) $\sigma([a, b]) = \sigma([1 - b, 1 - a])$ for any $0 \le a < b \le 1$;
(3) $\sigma([3a, 3b]) = 2\sigma([a, b])$ for any a, b such that $0 \le 3a < 3b \le 1$.

Complete the following with justification:

(1) Find $\sigma([0, 1/8])$.
(2) Calculate the second moment of σ, i.e. the integral

$$\int_0^1 x^2 \, d\sigma(x).$$

Exercise 57. Does the improper integral

$$\int_2^\infty \frac{x \sin(e^x)}{x + \sin(e^x)} \, dx$$

converge?

Exercise 58. Find the spectrum of the linear operator A in $L^2(\mathbb{R})$ defined as

$$(Af)(x) = \int_{-\infty}^{\infty} \frac{f(y)}{1 + (x - y)^2} \, dy.$$

(The spectrum of a linear operator T is the closure of the set of all complex numbers λ such that the operator $T - \lambda I$ does not have a bounded inverse. Hint: it may be helpful to find Fourier transform of $1/(1 + x^2)$.)

Exercise 59. Let $\{x_n\}_{n=1}^\infty$ be a sequence of elements in a Hilbert space H. Suppose that $x_n \to x \in H$ weakly in H and that $||x_n|| \to ||x||$ as $n \to \infty$. Show that then $||x_n - x|| \to 0$. Would the same be true for an arbitrary Banach space in place of H?

Exercise 60.

(1) Prove or disprove: there exists a distribution $u \in \mathcal{D}'(\mathbb{R})$ so that its restriction to $(0, \infty)$ is given by
$$\langle u, f \rangle = \int_0^\infty e^{1/x^2} f(x)\ dx$$
for all C^∞ which are compactly supported in $(0, \infty)$.

(2) Prove or disprove: there exists a distribution $u \in \mathcal{D}'(\mathbb{R})$ so that its restriction to $(0, \infty)$ is given by
$$\langle u, f \rangle = \int_0^\infty x^{-2} e^{i/x^2} f(x)\ dx$$
for all C^∞ which are compactly supported in $(0, \infty)$.

Exercise 61. Determine if
$$\sum_{n=1}^{\infty} \frac{\cos(k)}{k}$$
converges.

Exercise 62. For a Lebesgue measurable subset E of $\mathbb{R}$, denote by χ_E the indicator function of E. Let $\{E_n : n \in \mathbb{N}\}$ be a family of Lebesgue measurable subsets of $\mathbb{R}$ with finite measure and let f be a measurable function such that
$$\lim_{n\to\infty} \int_{\mathbb{R}} |f(x) - \chi_{E_n}|\ dx = 0.$$
Prove that f is almost everywhere equal to the indicator function of a measurable set.

Exercise 63. For a Lebesgue measurable subset E of $\mathbb{R}$, denote by χ_E the indicator function of E. Let $\{E_n : n \in \mathbb{N}\}$ be a family of Lebesgue measurable subsets of $\mathbb{R}$ with finite measure and let f be a measurable function such that
$$\lim_{n\to\infty} \int_{\mathbb{R}} |f(x) - \chi_{E_n}|\ dx = 0.$$
Prove that f is almost everywhere equal to the indicator function of a measurable set.

You've seen this problem before, but I'd invite you to think about a very short proof using some facts from modes of convergence.

Exercise 64. Let f be a C^1 function on $[0, \infty)$. Suppose that
$$\int_0^\infty t|f'(t)|^2\ dt < \infty,$$
$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(t)\ dt = L.$$
Show that $f(t) \to L$ as $t \to \infty$.

Exercise 65. Let K be a continuous function on $[0, 1] \times [0, 1]$ satisfying $|K| < 1$. Suppose that g is a continuous function on $[0, 1]$. Show that there exists a continuous function f on $[0, 1]$ such that
$$f(x) = g(x) + \int_0^1 f(y) K(x, y)\ dy.$$

Exercise 66. For $a, b \geq 0$, let

$$F(a,b) = \int_{-\infty}^{\infty} \frac{dx}{x^4 + (x-a)^4 + (x-b)^4}.$$

For what values of $p \in (0, \infty)$ is

$$\int_0^1 \int_0^1 F(a,b)^p \ da \ db < \infty?$$

Hint: try to prove that when $a \leq b$, $b^{-3}c \leq F(a,b) \leq b^{-3}C$ for positive constant $c < C$.

Exercise 67. Let $f : \mathbb{R} \to \mathbb{R}$ be a compactly supported function that satisfies the Hölder condition with exponent $\beta \in (0,1)$, that is, there exists a constant $A < \infty$ such that for all $x, y \in \mathbb{R}$, $|f(x) - f(y)| \leq A|x-y|^\beta$. Consider the function g defined by

$$g(x) = \int_{-\infty}^{\infty} \frac{f(y)}{|x-y|^\alpha} \ dy$$

where $\alpha \in (0, \beta)$

(1) Prove that g is a continuous function at 0.
(2) Prove that g is differentiable at 0. (Hint: Try the dominated convergence theorem.)

Exercise 68. A real-valued function f defined on $\mathbb{R}$ belongs to the space $C^{1/2}(\mathbb{R})$ if and only if

$$\sup_{x \in \mathbb{R}} |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{\sqrt{|x-y|}} < \infty.$$

Prove that a function $f \in C^{1/2}(\mathbb{R})$ if and only if there is a constant C so that for every $\varepsilon > 0$, there is a bounded function $\phi \in C^\infty(\mathbb{R})$ such that

$$\sup_{x \in \mathbb{R}} |f(x) - \phi(x)| \leq C\sqrt{\varepsilon} \text{ and } \sup_{x \in \mathbb{R}} \sqrt{\varepsilon}|\phi'(x)| \leq C.$$

Exercise 69. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Let $b_n \in \mathbb{R}$ be an increasing sequence with $\lim_{n\to\infty} b_n = \infty$. Show that

$$\lim_{n \to \infty} \frac{1}{b_n} \sum_{k=1}^{n} b_k a_k = 0.$$

Exercise 70. Let $L : [0,1] \to [0,1]$ be a function satisfying

$$|L(x_2) - L(x_1)| \leq |x_2 - x_1|/4, |L(1/2) - 1/2| < 1/4.$$

Prove that there is a continuous function $f : [0,1] \to [0,1]$ satisfying

$$f(x) = (1-x)L(f(x)) + 1/100.$$

Exercise 71. Show that $\int_0^\infty \frac{\sin(x)}{x^{2/3}} \ dx$ converges. Determine if

$$\int_1^\infty \frac{\sin(x)}{x^{2/3} + \sin(x)} \ dx$$

converges. *Hint: Use Taylor expansion.*

Exercise 72. Let $E \subset [0,1]$ be a measurable set with positive Lebesgue measure. Morever, assume it satisfied the following property: as long as x and y belong to E, we know $\frac{x+y}{2}$ belongs to E. Prove that E is an interval.

Exercise 73. Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of Lebesgue measurable functions such that f_n converges to f a.e. on $[0,1]$ and such that $||f_n||_{L^2[0,1]} \leq 1$ for all n. Show that

$$\lim_{n\to\infty} ||f_n - f||_{L^1([0,1])} = 0.$$

Exercise 74. A Hamel basis for a vector space X is a collection $\mathcal{H} \subset X$ of vectors such that each $x \in X$ can be written uniquely as a finite linear combination of elements in $\mathcal{H}$. Prove that an infinite dimensional Banach space cannot have a countable Hamel basis. *Hint: Otherwise the Banach space would be first category in itself.*

Exercise 75.

(1) Suppose Λ is a distribution on $\mathbb{R}^n$ such that $\operatorname{supp}(\Lambda) = \{0\}$. If $f \in C_c^\infty(R^n)$ satisfies $f(0) = 0$, does it follow that the product $f\Lambda = 0$ as a distribution?

(2) Suppose Λ is a distribution on $\mathbb{R}^n$ such that $\operatorname{supp}(\Lambda) \subset K$, where $K = \{x \in \mathbb{R}^n : |x| \leq 1\}$. If $f \in C_c^\infty(\mathbb{R})$ vanishes on K, does it follow that $f\Lambda = 0$ as a distribution?

Exercise 76. Show that $\ell^1(\mathbb{N}) \subsetneq (\ell^\infty)^*(\mathbb{N})$. *Hint*: Consider the sequence of averages

$$\phi_n(x) = \frac{1}{n}\sum_{j=1}^{n} x_j, \quad x = (x_1, x_2, \dots) \in \ell^\infty(\mathbb{N}).$$

Show that $\phi_n \in (\ell^\infty(\mathbb{N}))^*$ and consider its weak-* limit points.

Exercise 77. Let $K \subset \mathbb{R}^d$ be compact and let μ be a regular Borel measure on K with $\mu(K) = 1$. Prove that there exists a compact set $K_0 \subset K$ such that $\mu(K_0) = 1$ but $\mu(H) < 1$ for every compact $H \subsetneq K_0$.

Exercise 78. Assume that for every $x \in (0,1)$, the function f is absolutely continuous on $[0,x]$ and bounded variation on $[x,1]$. Assume also that f is continuous at 1. Prove that f is absolutely continuous on $[0,1]$.

Exercise 79. Let $H^1([0,1]) = \{f \in L^2([0,1]) : f' \in L^2\}$, where f' denotes the distributional derivative of f. Equip H^1 with the norm $||f||_{H^1} = ||f||_{L^2} + ||f'||_{L^2}$.

For $\alpha \in [0,1]$, denote $||f||_{C^\alpha} = \sup_{x\in[0,1]} |f(x)| + \sup_{x\neq y\in[0,1]} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$ and $C^\alpha([0,1]) = \{f \in C([0,1]) : ||f||_{C^\alpha} < \infty\}$.

You may use without proof that H^1 and C^α are both Banach spaces.

(1) Prove that $H^1([0,1]) \subset C^{1/2}([0,1])$.
(2) Prove that the closed unit ball in $H^1([0,1])$ is compact in $C^\alpha([0,1])$ for any $\alpha < 1/2$.
(3) Is the closed unit ball in $H^1([0,1])$ compact in $C^{1/2}([0,1])$? Prove or give a counterexample.

Exercise 80. *Extra 721 Problem:*

For $f \in L^2(\mathbb{R}^+)$, define $Tf(x) = \int_0^\infty \frac{f(y)}{x+2y}\, dy$. Prove that T is a bounded operator $L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$.

Exercise 81. *Extra 721 Problem:*

Let $U = \{x \in \mathbb{R}^n : |x| < 1\}$ be the open unit ball in $\mathbb{R}^n$. Let $\rho : U \to \mathbb{R}$ be a smooth function such that $\rho(0) = 0, \nabla\rho(0) \neq 0$. Let $\Sigma = \{x \in U \mid \rho(x) = 0\}$. For $x \in U$, let $d(x) = \inf_{y\in\Sigma} |x - y|$.

(1) For $x \in V = \{x \in \mathbb{R}^n : |x| \leq 1/2\}$, prove that there is a point $y \in \Sigma$ such that $d(x) = |x - y|$.
(2) For $x \in V \setminus \Sigma$ and for any $y \in \Sigma$ such that $d(x) = |x - y|$, prove that the vector $\nabla\rho(y)$ is a scalar multiple of $x - y$.
(3) Prove that there is an open set W with $0 \in W \subset V$ and a C^∞ function $\varphi : W \to \mathbb{R}$ such that for all $x \in W$, $|\varphi(x)| = d(x)$.

AN: This is a pretty old qual problem and part 2 and 3 feel more geometric (i.e. closer to a 761 problem) than most analysis qual problems now. Both require the implicit/inverse function theorem, but no theory beyond that.

Exercise 82. *Extra 721 Problem:*

Consider a differentiable function $f : \mathbb{R} \to \mathbb{R}$.

(1) Suppose the second derivative of f exists at x_0 (but not necessarily anywhere else). Show that $\lim_{h\to 0} \frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h^2} = f''(x_0)$.
(2) Suppose $\lim_{h\to 0} \frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h^2}$ exists. Recall that we have define f to be a differentiable function. Is it true that the second derivative of f exists at x_0?

Exercise 83. *Extra 721 Problem:*

Show that there is no sequence $\{a_n\}_{n\in\mathbb{N}}$ of positive numbers such that $\sum_{n\in\mathbb{N}} a_n|c_n| < \infty$ if and only if c_n is bounded.

Hint: Suppose such a sequence exists and consider the map $T : \ell^\infty(\mathbb{N}) \to \ell^1(\mathbb{N})$ *given by* $[Tf]_n = a_n f(n)$. *The set of* f *such that* $f(n) = 0$ *for all but finitely many* n *is dense in* ℓ^1 *but not in* ℓ^∞.

Exercise 84. *Extra 721 Problem:*

Let $C([0, 1])$ denote the set of continuous functions on $[0, 1]$ equipped with the sup-norm. Prove that there exists a dense subset of $C([0, 1])$ consisting of functions are nowhere differentiable.

Exercise 85. *Extra 721 Problem:*

Let H be a Hilbert space. For a linear space $Y \subset H$, define $Y^\perp = \{x \in H : (x, y) = 0\}$.

(1) Prove that if Y is closed, then $Y^\perp$ is a closed linear subspace of H.
(2) Prove that for any $x \in H$, a minimizing sequence for $\inf_{y\in Y} |x - y|$ is Cauchy. Conclude that we can uniquely write $x = x^{\|} + x^\perp$ with $x^{\|} \in Y$ and $x^\perp \in Y^\perp$.
(3) Prove that if $f : H \to \mathbb{R}$ is bounded and linear, then there exists $y \in H$ such that $f(x) = (x, y)$ for all x.

Exercise 86. *Extra 721 Problem:*

(1) Construct a set E such that on any interval non-empty finite interval I, $0 < |E \cap I| < |I|$.
(2) Prove or give a counterexample: there exists $\alpha \in (0, 1)$ and a measureable set E such that $\alpha|I| < |E \cap I| < |I|$ for every non-empty finite interval.

Exercise 87. *Extra 721 Problem:* Take a continuous function $K : [0,1]^2 \to \mathbb{R}$ and suppose $g \in C([0,1])$. Show that there exists a unique function $f \in C([0,1])$ such that

$$f(x) = g(x) + \int_0^x f(y)K(x,y)\ dy.$$

Exercise 88. *Extra 721 Problem:* Let f be a continuous real-valued function on $\mathbb{R}$ satisfying $|f(x)| \leq \frac{1}{1+x^2}$. Define F on $\mathbb{R}$ by

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n)$$

(a) Prove that F is continuous and periodic with period 1.
(b) Prove that if G is continuous and periodic with period 1, then

$$\int_0^1 F(x)G(x)\ dx = \int_{-\infty}^{\infty} f(x)G(x)\ dx.$$

Exercise 89. *Extra 721 Problem:* For $n \geq 2$ an integer, define $F(n)$ to be the function $F(n) = \max\{k \in \mathbb{Z} : 2^k/k \leq n\}$. Does $\sum_{n=2}^{\infty} 2^{-F(n)}$ converge?

Exercise 90. *Extra 721 Problem:* For a_n, b_n sequence in $\ell^2(\mathbb{N})$, prove that

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_n b_k}{n+k} \leq C||a||_2||b||_2.$$

Exercise 91. *Extra 721 Problem:* Let x_n be a sequence in a Hilbert space H. Suppose that x_n converges weakly to x as $N \to \infty$. Prove that there is a subsequence x_{n_k} such that

$$N^{-1}\sum_{k=1}^{N} x_{n_k}$$

converges in norm to x