DAY 9 PROBLEMS

Exercise 1. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set with finite Lebesgue measure $m(E)$. Define $f(r) = m((E+r) \cap E)$. Prove that f is continuous.

Exercise 2. Let $I = [a,b]$ and let $L^2(I)$ be the space of square-integrable functions on $[a,b]$ with scalar product $\langle f, g \rangle = \int_a^b f(t)\overline{g}(t)\, dt$. Let $p_0, p_1, \ldots$ be a sequence of real-valued polynomials p_n of degree exactly n such that

$$\int_a^b p_j(x)p_k(x)\, dx = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}.$$

Prove that $\{p_n\}_{n=0}^{\infty}$ is a *complete* orthonormal system.

Exercise 3. Let X, Y be σ-finite measure space with positive measures $d\mu, d\nu$ respectively and let K be a measurable function on $X \times Y$. Let u and w be nonnegative measurable functions on X, Y respectively. Assume that $u(x) > 0$ and $w(y) > 0$ a.e.. Suppose $1 < p < \infty$

$$\int_X |K(x,y)|u(x)d\mu(x) \leq Bw(y), \quad \nu - \text{a.e.}$$

$$\int_Y |K(x,y)|w(y)^{1/(p-1)}d\nu(y) \leq Au(x)^{1/(p-1)} \quad \mu - \text{a.e.}$$

Let T be defined by $Tf(x) = \int_Y K(x,y)f(y)\, d\nu(y)$. Show that T maps $L^p(Y)$ to $L^p(X)$ with operator norm bounded by $A^{1-1/p}B^{1/p}$.

Exercise 4. Suppose that $f \in L^1(\mathbb{R})$. Consider the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = \int_{\mathbb{R}} \frac{f(y)}{1+|xy|}\, dy.$$

(1) Prove that F is continuous,
(2) Prove that if $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, then $F \in L^p(\mathbb{R})$ for all $p > 2$.

Exercise 5. Let $E \subset [0,1]$ be a measurable set with positive Lebesgue measure. Morever, assume it satisfied the following property: as long as x and y belong to E, we know $\frac{x+y}{2}$ belongs to E. Prove that E is an interval.

Exercise 6. Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of Lebesgue measurable functions such that f_n converges to f a.e. on $[0,1]$ and such that $||f_n||_{L^2[0,1]} \leq 1$ for all n. Show that

$$\lim_{n\to\infty} ||f_n - f||_{L^1([0,1])} = 0.$$