DAY 8 PROBLEMS

Exercise 1. Suppose $f_n \in L^1(\mathbb{R})$ and the sequence ε_n of positive real numers satisfies $\lim_{n\to\infty} \varepsilon_n = 0$. Define the sets $E_n = \{x : |f_n(x)| \geq \varepsilon_n\}$ and assume that $\sum_n m(E_n) < \infty$. Prove that

(1) $\lim_{n\to\infty} f_n = 0$ Lebesgue a.e. on $\mathbb{R}$.
(2) For every $\delta > 0$, there is a set Ω such that $m(\Omega) < \delta$ and $\lim_{n\to\infty} f_n = 0$ uniformly on $\mathbb{R} \setminus \Omega$.

Exercise 2. Let $E \subset \mathbb{R}$. Suppose $g, f_n \in L^1(E)$, $\sup_n ||f_n||_{L^1} < \infty$ and $\lim_{n\to\infty} f_n = 0$ in Lebesgue measure. Prove that

$$\lim_{n\to\infty} \int_E \sqrt{|f_n g|}\, dx = 0.$$

Exercise 3. Suppose that on a set E of finite measure, $f_n \to f$ in measure and $g_n \to g$ in measure and f is finite a.e.. Prove that $f_n g_n \to fg$ in measure on E.

Hint: Can you prove $f_n^2 \to f^2$ in measure?

Exercise 4. Suppose f, f_n are Lebesgue measureable functions on $[0, 1]$ finite a.e.. Show that $f_n \to f$ in measure if and only if

$$\lim_{n\to\infty} \int_0^1 \frac{|f_n - f|(x)}{1 + |f_n - f|(x)}\, dx = 0.$$

Exercise 5. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable, real-valued functions on a measure space X such that $f_n \to f$ pointwise as $n \to \infty$, where $f : X \to \mathbb{R}$, and suppose that for some constant $M > 0$,

$$\int |f_n|\, d\mu \leq M \quad \text{for all } n \in \mathbb{N}.$$

(1) Prove that

$$\int |f|\, d\mu \leq M.$$

(2) Give an example to show that we may have $\int |f_n|\, d\mu = M$ for every $n \in \mathbb{N}$, but $\int |f|\, d\mu < M$.
(3) Prove that

$$\lim_{n\to\infty} \int ||f_n| - |f| - |f_n - f||\, d\mu = 0.$$

Exercise 6. Let $f \in L^1(\mathbb{R})$ satisfy $\int_a^b f(x)\, dx = 0$ for any two rational numbers $a < b$. Does it follow that $f(x) = 0$ for almost every x?

Exercise 7. For a Lebesgue measurable subset E of $\mathbb{R}$, denote by χ_E the indicator function of E. Let $\{E_n : n \in \mathbb{N}\}$ be a family of Lebesgue measurable subsets of $\mathbb{R}$ with finite measure and let f be a measurable function such that

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f(x) - \chi_{E_n}| \, dx = 0.$$

Prove that f is almost everywhere equal to the indicator function of a measurable set.

You've seen this problem before, but I'd invite you to think about a very short proof using some facts from modes of convergence.