DAY 7 PROBLEMS

Exercise 1. Prove that there does not exist $f \in L^1(\mathbb{R})$ such that $g * f(x) = g(x)$ for any $g \in C_0(\mathbb{R})$ and $x \in \mathbb{R}$.

Exercise 2. For any $n \geq 1$, show that there exists closed sets $A, B \subset \mathbb{R}^n$ with $|A| = |B| = 0$, but $|A + B| > 0$ (as usual $A + B = \{a + b : a \in A, b \in B\}$).

Exercise 3. Given $\alpha \geq 0$ the α-dimensional Hausdorff measure of a set $X \subset \mathbb{R}^n$ is

$$\mathcal{H}^\alpha(X) = \lim_{r \to 0} \inf\{\sum_{i=1}^{\infty} r_i^\alpha : X \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), r_i < r \text{ for all } i\}$$

and the Hausdorff dimension is $\dim_H(X) = \inf\{\alpha \geq 0 : \mathcal{H}^\alpha(X) = 0\}$.

Prove the following:

(1) If $X \subset \mathbb{R}^n$ and μ is a finite Borel measure on $\mathbb{R}^n$ such that $\mu(X) > 0$ and $\mu(B(x, r)) \leq r^\alpha$ for all open balls $B(x, r)$, then $\dim_H(X) \geq \alpha$.

(2) If $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is the unit circle, the $\dim_H(S^1) = 1$.

Exercise 4. Let $z_1, z_2, \ldots, z_n$ be points on the unit circle $\mathbf{T} = \{|z| = 1\}$ in the complex plane. Let $E \subset \mathbf{T}$ satisfy $m(E) > 2\pi(1 - \frac{1}{n})$. Prove that E can be rotated so that all the points z_k fall into the rotated set, i.e., that there exists $\alpha \in \mathbf{T}$ such that $\alpha z_k \in E$ for $k = 1, 2, \ldots, n$.

Exercise 5. Let σ be a Borel probability measure on $[0, 1]$ satisfying

(1) $\sigma([1/3, 2/3]) = 0$;

(2) $\sigma([a, b]) = \sigma([1 - b, 1 - a])$ for any $0 \leq a < b \leq 1$;

(3) $\sigma([3a, 3b]) = 2\sigma([a, b])$ for any a, b such that $0 \leq 3a < 3b \leq 1$.

Complete the following with justification:

(1) Find $\sigma([0, 1/8])$.

(2) Calculate the second moment of σ, i.e. the integral

$$\int_0^1 x^2 \, d\sigma(x).$$

Exercise 6. Assume that for every $x \in (0, 1)$, the function f is absolutely continuous on $[0, x]$ and bounded variation on $[x, 1]$. Assume also that f is continuous at 1. Prove that f is absolutely continuous on $[0, 1]$.