DAY 6 PROBLEMS

Exercise 1. Let $f_n : X \to \overline{\mathbb{R}}$ be a sequence of measurable functions on a finite measure space X, so that $|f_n(x)| < \infty$ for almost every $x \in X$. Show that there is a sequence A_n of positive real numbers so that

$$\lim_{n\to\infty} \frac{f_n(x)}{A_n} = 0$$

almost everywhere. *Hint: Borel-Cantelli.*

Exercise 2. Suppose $E \subset \mathbb{R}^d$ is a given set and O_n is the open set $O_n = \{x : d(x, E) < 1/n\}$.

(1) Show that if E is compact, then $|E| = \lim_{n\to\infty} |O_n|$.
(2) Is the statement false for E closed and unbounded?
(3) Is the statement false for E open and bounded?

Exercise 3. Let $X = [0, 1]$ with Lebesgue measure and $Y = [0, 1]$ with counting measure. Give an example of a measurable function $f : X \times Y \to [0, \infty)$ for which Fubini's theorem does not apply. (This example shows that the theorem is not valid if the hypothesis of σ-finiteness is omitted.)

Exercise 4. For a Lebesgue measurable subset E of $\mathbb{R}$, denote by χ_E the indicator function of E. Let $\{E_n : n \in \mathbb{N}\}$ be a family of Lebesgue measurable subsets of $\mathbb{R}$ with finite measure and let f be a measurable function such that

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f(x) - \chi_{E_n}| \, dx = 0.$$

Prove that f is almost everywhere equal to the indicator function of a measurable set.

Exercise 5. Let $K \subset \mathbb{R}^d$ be compact and let μ be a regular Borel measure on K with $\mu(K) = 1$. Prove that there exists a compact set $K_0 \subset K$ such that $\mu(K_0) = 1$ but $\mu(H) < 1$ for every compact $H \subsetneq K_0$.