DAY 5 PROBLEMS

Exercise 1. Consider the space $C([0,10])$ of continuous functions on $[0,10]$, and for a given large number L consider the metric $d_L(f,g) = \max_{x\in[0,10]} e^{-Lx}|f(x)-g(x)|$.

(1) Argue that $C([0,10])$ with the metric d_L is a complete metric space.
(2) Show that there is a unique function which is continuous on $[0,10]$ and satisfies

$$f(x) = -15 + \cos(x) \int_0^x e^{e^{tx}} f(t)\, dt$$

for all $x \in [0,10]$.

Exercise 2. Prove that there are two functions $f_1, f_2 \in C[0,1]$ that solve the following system of equations for all $x \in [0,1]$,

$$20f_1(x) + 3f_2(x) = \sin(x) + \int_0^1 \sin(xt)\sin(f_1(t))\, dt$$

$$-f_1(x) + 10f_2(x) = \cos(x) - \int_0^{1/2} \cos(xt)\cos(f_2(t))\, dt.$$

Exercise 3. Can one find a bounded sequence of real numbers $x_n, n \in \mathbb{Z}$ that satisfies

$$x_n = \sin(n) + 0.5x_{n-1} + 0.4\sin(x_{n+1})$$

for every $n \in \mathbb{Z}$?

Exercise 4. Consider the following equation for an unknown function $f : [0,1] \to \mathbb{R}$:

$$f(x) = g(x) + \lambda \int_0^1 (x-y)^2 f(y)\, dy + \frac{1}{2}\sin(f(x)).$$

Prove that there exists a number λ_0 such that for all $\lambda \in [0, \lambda_0)$ and all continuous functions g on $[0,1]$, the equation has a continuous solution.

Exercise 5. Let K be a continuous function on $[0,1]\times[0,1]$ satisfying $|K| < 1$. Suppose that g is a continuous function on $[0,1]$. Show that there exists a continuous function f on $[0,1]$ such that

$$f(x) = g(x) + \int_0^1 f(y)K(x,y)\, dy.$$

Exercise 6. Let $L : [0,1] \to [0,1]$ be a function satisfying

$$|L(x_2) - L(x_1)| \le |x_2 - x_1|/4, |L(1/2) - 1/2| < 1/4.$$

Prove that there is a continuous function $f : [0,1] \to [0,1]$ satisfying

$$f(x) = (1-x)L(f(x)) + 1/100.$$