DAY 4 PROBLEMS

Exercise 1. Let $K : [a,b] \times [a,b] \to \mathbb{R}$ be a differentiable function such that

$$\max_{[a,b]^2} |K(x,t)| \leq 1, \max_{[a,b]^2} \left| \frac{\partial K}{\partial x}(x,t) \right| \leq 1.$$

Consider the space $C[a,b]$ of continuous functions on $[a,b]$ with the sup-norm. For $f \in C[a,b]$, define

$$Af(x) = \int_a^b K(x,t) f(t) \, dt.$$

(1) Prove that $\{Af : \max_{[a,b]} |f(x)| \leq 1\}$ is a totally bounded subset of $C[a,b]$.
(2) If in (1) we drop the assumption $\max_{[a,b]^2} \left| \frac{\partial K}{\partial x}(x,t) \right| \leq 1$ and keep the other assumptions, does $\{Af : \max_{[a,b]} |f(x)| \leq 1\}$ have to be a totally bounded subset of $C[a,b]$?

Exercise 2. Let $K : [a,b] \times [a,b] \to \mathbb{R}$ be a continuous function. Consider the space $C[a,b]$ of continuous functions on $[a,b]$ with the sup-norm. For $f \in C[a,b]$, define

$$S_K f(x) = \int_a^b K(x,t) f(t) \, dt.$$

(1) Is $\{S_K f : \max_{[a,b]} |f(x)| \leq 1\}$ necessarily a totally bounded subset of $C[a,b]$?
(2) Let f_n be a sequence of continuous functions on $[a,b]$ satisfying

$$\sup_n \sup_{x \in [a,b]} |f_n(x)| \leq 1.$$

Does the sequence $S_K f_n$ necessarily have a convergent subsequence? Give a proof or counterexample.
(3) Let K_n be a sequence of continuous functions on $[a,b] \times [a,b]$ and assume that

$$\sup_n \max\{|K_n(x,y)| : (x,y) \in [a,b] \times [a,b]\} \leq 1.$$

Let $f \in C[a,b]$. Does the sequence $S_{K_n} f$ necessarily have a convergent subsequence in $C[a,b]$? Prove or give a counterexample.

Exercise 3. For $f \in L^2$, let $F(x) = \int_0^x f(x) \, dx$.
(1) Prove that

$$\int_0^1 \left(\frac{F(x)}{x} \right)^2 dx \leq 4 \int_0^1 f^2(x) \, dx.$$

(2) Define

$$Af(x) = \frac{1}{x\sqrt{1 + |\log(x)|}} \int_0^x f(t) \, dt.$$

Prove that if f_n is a sequence of continuous functions with $\sup_n ||f_n||_{L^2([0,1])} \leq 1$, then Af_n has a subsequence converging in the L^2 norm.

Exercise 4. Suppose S is the set of real-valued functions continuous g on $[0,1]$ that satisfy two conditions:

$$\left|\int_0^1 g(x)\ dx\right| \leq 1$$

and

$$|g(x) - g(y)| \leq |x-y|^{1/2}$$

for each $x, y \in [0,1]$. Consider the functional

$$F(g) = \int_0^1 (1-5x^2)g^{10}(x)\ dx.$$

Is F bounded on S? Does it acheive it's maximum on S?

Exercise 5. Suppose that $f_n : [0,1] \to \mathbb{R}$ is a sequence of continuous functions each of which has continuous first and second derivatives on $(0,1)$. Prove: If

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \text{for all} \quad x \in [0,1]$$

and

$$\sup_{n\geq 1} \max_{0<x<1} |f_n''(x)| < \infty,$$

then f' exists and is continuous on $(0,1)$.