DAY 3 PROBLEMS

Exercise 1. Take $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^n$ and let $X + Y = \{x + y : x \in X, y \in Y\}$.

(1) Assume X is closed and Y is compact. Prove that $X + Y$ is closed.
(2) If Y is closed but not compact, is $X + Y$ closed? Prove or give a counterexample.

Exercise 2. A function $f : U \to \mathbb{R}$ defined on a subset $U \subset \mathbb{R}^n$ is

- *locally bounded* if for all $x \in U$ there exists $\varepsilon, R > 0$ such that $|f(y)| \leq R$ for all $y \in U$ with $|x - y| < \varepsilon$,
- *globally bounded* if there exists $R > 0$ such that $|f(y)| \leq R$ for all $y \in U$.

Prove: If $U \subset \mathbb{R}^n$, then the following are equivalent:

(1) U is compact,
(2) every locally bounded function $f : U \to \mathbb{R}$ is globally bounded.

Exercise 3. Let $\mathcal{K}$ denote the collection of compact subsets of $[0, 1]$. Define the Hausdorff metric on $\mathcal{K}$ by

$$d(K_1, K_2) = \sup_{x \in K_1} \inf_{y \in K_2} |x - y| + \sup_{x \in K_2} \inf_{y \in K_1} |x - y|.$$

Prove that $(\mathcal{K}, d)$ is a complete metric space.

Exercise 4. Prove that any open set $U \subset \mathbb{R}^n$ can be expressed as a countable union of rectangles.

Exercise 5. Let $x_1, \ldots, x_{n+1}$ be pairwise distinct real numbers. Prove that there exists $C > 0$ such that: if $P : \mathbb{R} \to \mathbb{R}$ is a polynomial with degree at most n, then

$$\max_{x \in [0,1]} |P(x)| \leq C \max\{|P(x_1)|, \ldots, |P(x_{n+1})|\}.$$

Exercise 6. Let $f \in C^1([0, 1])$. Show that for every $\varepsilon > 0$ there exists a polynomial p such that

$$||f - p||_\infty + ||f' - p'||_\infty < \varepsilon.$$

Exercise 7. Let $a = (a_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers. Prove that the set

$$X = \{(x_n)_{n \in \mathbb{N}} \in \ell^1(\mathbb{N}) : x_n \in [0, a_n] \text{ for all } n \in \mathbb{N}\}$$

is compact in the $\ell^1(\mathbb{N})$ norm if and only if $(a_n)_{n \in \mathbb{N}} \in \ell^1(\mathbb{N})$.