DAY 2 PROBLEMS

Exercise 1. Let $\{a_n\}$ be a convergent sequence of complex numbers and let $\lim_{n\to\infty} a_n = L$. Let

$$c_n := \frac{1}{n^5}\sum_{k=1}^{n} k^4 a_k.$$

Prove that c_n converges and determine its limit.

Exercise 2. Does the improper integral

$$\int_2^\infty \frac{x\sin(e^x)}{x+\sin(e^x)}\,dx$$

converge?

Exercise 3. Let f be a C^1 function on $[0,\infty)$. Suppose that

$$\int_0^\infty t|f'(t)|^2\,dt < \infty,$$

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(t)\,dt = L.$$

Show that $f(t)\to L$ as $t\to\infty$.

Exercise 4. Let $f:\mathbb{R}\to\mathbb{R}$ be a compactly supported function that satisfies the Hölder condition with exponent $\beta\in(0,1)$, that is, there exists a constant $A<\infty$ such that for all $x,y\in\mathbb{R}$, $|f(x)-f(y)|\le A|x-y|^\beta$. Consider the function g defined by

$$g(x) = \int_{-\infty}^\infty \frac{f(y)}{|x-y|^\alpha}\,dy$$

where $\alpha\in(0,\beta)$

(1) Prove that g is a continuous function at 0.
(2) Prove that g is differentiable at 0. (Hint: Try the dominated convergence theorem.)

Exercise 5. Show that $\int_0^\infty \frac{\sin(x)}{x^{2/3}}\,dx$ converges. Determine if

$$\int_1^\infty \frac{\sin(x)}{x^{2/3}+\sin(x)}\,dx$$

converges. *Hint: Use Taylor expansion.*