DAY 15 PROBLEMS

Exercise 1. For $s > \frac{1}{2}$, let $H^s(\mathbb{R}^n)$ denote the Sobolev space

$$H^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, d\mu(\xi) < \infty\}$$

(where μ is the Lebesgue measure and $\hat{f}$ is the Fourier transform of f). Show that if $u, v \in H^s(\mathbb{R}^n)$ for $s > n/2$, the $uv \in H^s(\mathbb{R}^n)$ and

$$||uv||_{H^s(\mathbb{R}^n)} \leq C||u||_{H^s(\mathbb{R}^n)}||v||_{H^s(\mathbb{R}^n)}$$

for a constant C depending only on s and n.

Exercise 2. For $s > \frac{1}{2}$ let $H^s(\mathbb{R}^n)$ denote the Sobolev space

$$H^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, d\mu(\xi) < +\infty\}$$

(where μ is the Lebesgue measure and $\hat{f}$ is the Fourier transform of f). Use the Fourier transform to prove that if $u \in H^s(\mathbb{R}^n)$ for $s > n/2$, then $u \in L^\infty(\mathbb{R}^n)$, wih the bound

$$||u||_{L^\infty} \leq C||u||_{H^s(\mathbb{R}^n)}$$

for a constant C depending only on s and n.

Exercise 3. Let $H^s(\mathbb{R})$ be the Sobolev space on $\mathbb{R}$ with the norm

$$||u||^2_{H^s} = \int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi.$$

Prove that for non-negative real numbers $r < s < t$, for any $\varepsilon > 0$, there exists $C > 0$ such that

$$||u||_{H^s} \leq \varepsilon||u||_{H^t} + C||u||_{H^r} \text{ whenever } u \in H^t(\mathbb{R}).$$

Exercise 4. *Extra 721 Problem:*
Prove that if K is a subset of $\mathbb{R}^n$ such that every continuous real-valued function on K is bounded, then K is compact.

Exercise 5. *Extra 721 Problem:*
Let $f : [0, 1] \to \mathbb{R}$ be continuous with $\min_{0 \leq x \leq 1} f(x) = 0$. Assume that for all $0 \leq a \leq b \leq 1$, we have $\int_a^b [f(x) - \min_{a \leq y \leq b} f(y)] \, dx \leq \frac{|b-a|}{2}$. Prove that for all $\lambda \geq 0$, we have

$$|\{x : f(x) > \lambda + 1\}| \leq \tfrac{1}{2}|\{x : f(x) > \lambda\}|.$$

Exercise 6. *Extra 721 Problem:*
Consider the sequence of function $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \int_0^n \frac{\sin(sx)}{\sqrt{s}}\, ds.$$

(a) Show that f_n converges uniformly as $n \to \infty$ on any interval (α, β) for $0 < \alpha < \beta < \infty$.
(b) Show that f_n does not converge uniformly on $(0, 1]$ as $n \to \infty$.
(c) Does f_n converge uniformly on $[1, \infty)$ as $n \to \infty$?

Exercise 7. *Extra 721 Problem:* For $c_k \in \mathbb{R}$, say that $\prod_k c_k$ convergences if $\lim_{K\to\infty} \prod_{k=1}^K c_k = C$ exists for $C \neq 0, \infty$.

(a) Prove that if $0 < a_k < 1$ for all k, or if $-1 < a_k < 0$, for all k, then $\prod_k (1 + a_k)$ converges if and only if $\sum_k a_k$ conveges.
(b) However, prove that $\prod_k (1 + \frac{(-1)^k}{\sqrt{k}})$ diverges.