DAY 13 PROBLEMS

Exercise 1. The following distributions u, v on $\mathbb{R}^2$ are defined by pairing with Schwartz functions via

$$\langle u, \phi \rangle = \int_0^2 \phi(0,t)\ dt$$

$$\langle v, \phi \rangle = \int_0^2 \phi(t,0)\ dt$$

Show that the convolution $u * v$ can be identified with a finite, absolutely continuous measure μ. Find $g \in L^1(\mathbb{R}^2)$ such that $\int \phi\ d\mu = \int \phi g\ dx$ for all Schwartz functions ϕ.

Exercise 2. Let $\mathcal{D}'(\mathbb{R})$ denote the space of distributions on $\mathbb{R}$ with the weak-$*$ topology. Determine the limit in $\mathcal{D}'(\mathbb{R})$ of the sequence of functions in $\mathbb{R}$:

$$\lim_{n\to\infty} \sqrt{n} e^{\frac{i}{2} n x^2}.$$

Exercise 3. A real-valued function f defined on $\mathbb{R}$ belongs to the space $C^{1/2}(\mathbb{R})$ if and only if

$$\sup_{x\in\mathbb{R}} |f(x)| + \sup_{x\neq y} \frac{|f(x)-f(y)|}{\sqrt{|x-y|}} < \infty.$$

Prove that a function $f \in C^{1/2}(\mathbb{R})$ if and only if there is a constant C so that for every $\varepsilon > 0$, there is a bounded function $\phi \in C^\infty(\mathbb{R})$ such that

$$\sup_{x\in\mathbb{R}} |f(x) - \phi(x)| \leq C\sqrt{\varepsilon} \text{ and } \sup_{x\in\mathbb{R}} \sqrt{\varepsilon}|\phi'(x)| \leq C.$$

Exercise 4. Let $H^1([0,1]) = \{f \in L^2([0,1]) : f' \in L^2\}$, where f' denotes the distributional derivative of f. Equip H^1 with the norm $||f||_{H^1} = ||f||_{L^2} + ||f'||_{L^2}$.

For $\alpha \in [0,1]$, denote $||f||_{C^\alpha} = \sup_{x\in[0,1]} |f(x)| + \sup_{x\neq y\in[0,1]} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$ and $C^\alpha([0,1]) = \{f \in C([0,1]) : ||f||_{C^\alpha} < \infty\}$.

(1) Prove that $H^1([0,1]) \subset C^{1/2}([0,1])$.
(2) Prove that the closed unit ball in $H^1([0,1])$ is compact in $C^\alpha([0,1])$ for any $\alpha < 1/2$.
(3) Is the closed unit ball in $H^1([0,1])$ compact in $C^{1/2}([0,1])$? Prove or give a counterexample.

Exercise 5. *Extra 721 Problem:*

Prove that there exists a constant $C > 0$ such that if $\mathcal{B}$ be a collection of closed balls in $\mathbb{R}^2$ with positive radius and centers in a compact set K, then there exists (possibly empty) subcollections $\mathcal{B}_1, \ldots, \mathcal{B}_C$ of $\mathcal{B}$ such that the elements $\mathcal{B}_i$ are disjoint and $\{x : x \text{ is a center of } B \text{ for some } B \in \mathcal{B}\} \subset \bigcup_{i=1}^C \bigcup_{B\in\mathcal{B}_i} B$.

Exercise 6. *Extra 721 Problem:*

Let $f : \mathbb{R} \to \mathbb{R}$ be smooth and suppose for each $x \in \mathbb{R}$, there exists $n(x)$ such that $f^{(n(x))}(x) = 0$. Prove that f is a polynomial.

Exercise 7. *Extra 721 Problem:* Let H be a Hilbert space. For a linear space $Y \subset H$, define $Y^\perp = \{x \in H : (x, y) = 0\}$.

(1) Prove that if Y is closed, then $Y^\perp$ is a closed linear subspace of H.
(2) Prove that if $f : H \to \mathbb{R}$ is bounded and linear, then there exists $y \in H$ such that $f(x) = (x, y)$ for all x.