DAY 12 PROBLEMS

Exercise 1. Prove that there is a distribution $u \in \mathcal{D}'(\mathbb{R})$ so that its restriction to $(0,\infty)$ is given by

$$\langle u, f\rangle = \int_0^\infty x^{-2}\cos(x^{-2})f(x)\,dx$$

for all $f \in C^\infty(\mathbb{R})$ compactly supported on $(0,\infty)$ and $\langle u, f\rangle = 0$ for all $f \in C^\infty(\mathbb{R})$ compactly supported on $(-\infty, 0)$.

Exercise 2. On $\mathbb{R}\setminus\{0\}$ define $f(x) = |x|^{-7/2}$. Find a tempered distribution $h \in \mathcal{S}'(\mathbb{R})$ so that $f = h$ on $\mathbb{R}\setminus\{0\}$.

Exercise 3.

(1) Prove or disprove: there exists a distribution $u \in \mathcal{D}'(\mathbb{R})$ so that its restriction to $(0,\infty)$ is given by

$$\langle u, f\rangle = \int_0^\infty e^{1/x^2} f(x)\,dx$$

for all C^∞ which are compactly supported in $(0,\infty)$.

(2) Prove or disprove: there exists a distribution $u \in \mathcal{D}'(\mathbb{R})$ so that its restriction to $(0,\infty)$ is given by

$$\langle u, f\rangle = \int_0^\infty x^{-2}e^{i/x^2} f(x)\,dx$$

for all C^∞ which are compactly supported in $(0,\infty)$.

Exercise 4.

(1) Suppose Λ is a distribution on $\mathbb{R}^n$ such that $\operatorname{supp}(\Lambda) = \{0\}$. If $f \in C_c^\infty(R^n)$ satisfies $f(0) = 0$, does it follow that the product $f\Lambda = 0$ as a distribution?

(2) Suppose Λ is a distribution on $\mathbb{R}^n$ such that $\operatorname{supp}(\Lambda) \subset K$, where $K = \{x \in \mathbb{R}^n : |x| \le 1\}$. If $f \in C_c^\infty(\mathbb{R})$ vanishes on K, does it follow that $f\Lambda = 0$ as a distribution?

Exercise 5. *Extra 721 Problem:*

For $f \in L^2(\mathbb{R}^+)$, define $Tf(x) = \int_0^\infty \frac{f(y)}{x+2y}\,dy$. Prove that T is a bounded operator $L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$.

Exercise 6. *Extra 721 Problem:*

Let $U = \{x \in \mathbb{R}^n : |x| < 1\}$ be the open unit ball in $\mathbb{R}^n$. Let $\rho : U \to \mathbb{R}$ be a smooth function such that $\rho(0) = 0, \nabla\rho(0) \neq 0$. Let $\Sigma = \{x \in U \mid \rho(x) = 0\}$. For $x \in U$, let $d(x) = \inf_{y\in\Sigma}|x - y|$.

(1) For $x \in V = \{x \in \mathbb{R}^n : |x| \le 1/2\}$, prove that there is a point $y \in \Sigma$ such that $d(x) = |x - y|$.

(2) For $x \in V \setminus \Sigma$ and for any $y \in \Sigma$ such that $d(x) = |x - y|$, prove that the vector $\nabla\rho(y)$ is a scalar multiple of $x - y$.

(3) Prove that there is an open set W with $0 \in W \subset V$ and a C^∞ function $\varphi : W \to \mathbb{R}$ such that for all $x \in W$, $|\varphi(x)| = d(x)$.

AN: This is a pretty old qual problem and part 2 and 3 feel more geometric (i.e. closer to a 761 problem) than most analysis qual problems now. Both require the implicit/inverse function theorem, but no theory beyond that.

Exercise 7. *Extra 721 Problem:*

Consider a differentiable function $f : \mathbb{R} \to \mathbb{R}$.

(1) Suppose the second derivative of f exists at x_0 (but not necessarily anywhere else). Show that $\lim_{h\to 0} \frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h^2} = f''(x_0)$.

(2) Suppose $\lim_{h\to 0} \frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h^2}$ exists. Recall that we have define f to be a differentiable function. Is it true that the second derivative of f exists at x_0?