DAY 11 PROBLEMS

Exercise 1. Let $X = \{P : \mathbb{R} \to \mathbb{R} | P \text{ is a polynomial}\}$. Prove that there does not exist a norm $|| \cdot ||$ on X such that $(X, || \cdot ||)$ is a Banach space.

Exercise 2. For $f, g \in L^2[0,1]$, let $\langle f, g \rangle = \int_0^1 f(x)\overline{g}(x)\, dx$ and set

$$g_n(x) = \frac{n^{2/3} \sin(n/x)}{xn + 1}.$$

Does there exists $\alpha > 0$ such that

$$\sum_{n=1}^{\infty} |\langle f, g_n \rangle|^{\alpha} < \infty.$$

hold for every $f \in L^2$? *Hint: is $||g_n||_{L^2}$ a bounded sequence?*

Exercise 3. Let f_n be a sequence of continuous functions on $I = [0,1]$. Suppose that for every $x \in I$ there exists an $M(x) < \infty$ so that $|f_n(x)| \leq M(x)$ for all $n \in \mathbb{N}$. Show then that $\{f_n\}$ is uniformly bounded on some interval, that is there exists $M \in \mathbb{R}$ and an interval $(a,b) \subset I$ so that $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and $x \in (a,b)$.

Exercise 4. Assume that X is a compact metric space and $T : X \to X$ is a continuous map. Let $\mathcal{M}_1(T)$ denote the set of Borel probability measures on X such that $T_*\mu = \mu$. Prove:

1. $\mathcal{M}_1(T) \neq \emptyset$.
2. If $\mathcal{M}_1(T) = \{\mu\}$ consists of a single measure μ, then

$$\int_X f\, d\mu = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n(x)$$

for every continuous function $f : X \to \mathbb{R}$ and point $x \in X$.

Exercise 5. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of elements in a Hilbert space H. Suppose that $x_n \to x \in H$ weakly in H and that $||x_n|| \to ||x||$ as $n \to \infty$. Show that then $||x_n - x|| \to 0$. Would the same be true for an arbitrary Banach space in place of H?

Exercise 6. A Hamel basis for a vector space X is a collection $\mathcal{H} \subset X$ of vectors such that each $x \in X$ can be written uniquely as a finite linear combination of elements in $\mathcal{H}$. Prove that an infinite dimensional Banach space cannot have a countable Hamel basis. *Hint: Otherwise the Banach space would be first category in itself.*

Exercise 7. Show that $\ell^1(\mathbb{N}) \subsetneq (\ell^{\infty})^*(\mathbb{N})$. *Hint*: Consider the sequence of averages

$$\phi_n(x) = \frac{1}{n} \sum_{j=1}^{n} x_j, \quad x = (x_1, x_2, \dots) \in \ell^{\infty}(\mathbb{N}).$$

Show that $\phi_n \in (\ell^{\infty}(\mathbb{N}))^*$ and consider its weak-* limit points.

Exercise 8. *Extra 721 Problem:*

(1) Construct a set E such that on any interval non-empty finite interval I, $0 < |E \cap I| < |I|$.

(2) Prove or give a counterexample: there exists $\alpha \in (0, 1)$ and a measureable set E such that $\alpha|I| < |E \cap I| < |I|$ for every non-empty finite interval.

Exercise 9. *Extra 721 Problem:* Take a continuous function $K : [0, 1]^2 \to \mathbb{R}$ and suppose $g \in C([0, 1])$. Show that there exists a unique function $f \in C([0, 1])$ such that

$$f(x) = g(x) + \int_0^x f(y)K(x, y)\ dy.$$

Exercise 10. *Extra 721 Problem:* Let f be a continuous real-valued function on $\mathbb{R}$ satisfying $|f(x)| \leq \frac{1}{1+x^2}$. Define F on $\mathbb{R}$ by

$$F(x) = \sum_{n=-\infty}^{\infty} f(x + n)$$

(a) Prove that F is continuous and periodic with period 1.

(b) Prove that if G is continuous and periodic with period 1, then

$$\int_0^1 F(x)G(x)\ dx = \int_{-\infty}^{\infty} f(x)G(x)\ dx.$$