DAY 10 PROBLEMS

Exercise 1. Suppose that E is a measurable set of real numbers with arbitrarily small periods (that is, there exists a sequence of real numbers $p_i \to 0$ such that $E + p_i = E$). Prove that either E or it's complement has measure 0.

Exercise 2. Let $\mathbf{T} \subset \mathbb{C}$ be the unit circle. We say that $G \subset \mathbf{T}$ is a subgroup of $\mathbf{T}$ if $1 \in G$, $\zeta_1, \zeta_2 \in G$ implies $\zeta_1\zeta_2 \in G$ and $\zeta_1 \in G$ implies $\zeta_1^{-1} \in G$.

(1) What are the compact subgroups of $\mathbf{T}$?
(2) Give an example of an infinite subgroup $G \subsetneq \mathbf{T}$.
(3) Prove or give a counterexample: there are no measurable subgroups $G \subsetneq \mathbf{T}$ with $|G| > 0$.

Exercise 3. Let $I = [0, 1)$ and given $N \in \mathbb{N}$ consider the dyadic intervals $I_{j,N} = [j2^{-N}, (j+1)2^{-N})$ for $j \in \{0, 1, \ldots, 2^N - 1\}$. For a function $f \in L^1(I)$, define a sequence of function $E_N f : I \to \mathbb{R}$ by

$$E_N f(x) = 2^N \int_{I_{j,N}} f(t)\ dt \qquad \text{for } x \in I_{j,N}.$$

Show that $\lim_{N\to\infty} E_N f(x) = x$ for a.e. $x \in I$.

Exercise 4. For $x, y \in \mathbb{R}$, let $K(y) = \pi^{-1}(1 + y^2)^{-1}$, and for $t > 0$ let

$$P_t f(x) = \int_{-\infty}^{\infty} t^{-1} K(t^{-1} y) f(x - y)\ dy.$$

(1) Show that if f is continuous and compactly supported, then

$$\lim_{t\to 0^+} \sup_{x\in\mathbb{R}} |P_t f(x) - f(x)| = 0.$$

(2) Let $p \geq 1$. For $f \in L^p(\mathbb{R})$ denote by Mf the Hardy-Littlewood maximal function of f. Show that there is a constant $C > 0$ so that for all $f \in L^p(\mathbb{R})$ the inequality

$$|P_t f(x)| \leq CMf(x)$$

holds for every $x \in \mathbb{R}$ and every $t > 0$.

(3) If $f \in L^1(\mathbb{R})$, prove that $\lim_{t\to 0+} P_t f(x) = f(x)$ for almost every $x \in \mathbb{R}$.

Exercise 5. Given a real number x, let $\{x\}$ denote the fractional part of x. Suppose α is an irrational number and define $T : [0, 1] \to [0, 1]$ by

$$T(x) = \{x + \alpha\}.$$

Prove: If $A \subset [0, 1]$ is measurable and $T(A) = A$, then $|A| \in \{0, 1\}$.