DAY 1 PROBLEMS

Exercise 1. For a sequence (a_k) let $s_n = \sum_{k=1}^n a_k$ and $\sigma_L = \frac{1}{L}\sum_{n=1}^L s_n$. We say that $\sum_{k=1}^\infty a_k$ is Cesáro summable to S if $\lim_{L\to\infty} \sigma_L = S$.

(1) Prove: $s_n - \sigma_n = \frac{(n-1)a_n + (n-2)a_{n-1} + \cdots + a_2}{n}$.
(2) Prove: If $\sum_{k=1}^\infty a_k$ is Cesáro summable to S and if $\lim_{k\to\infty} k a_k = 0$, then $\sum_{k=1}^\infty a_k$ converges and $\sum_{k=1}^\infty a_k = S$.

Exercise 2. Let

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < x_1^2 \leq 1/2\}.$$

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x) = (x_1^2 + x_2^2)^{-b/2} |\log(x_1^2 + x_2^2)|^{-\gamma}.$$

Determine for which values $b > 0, \gamma \in \mathbb{R}$, $\int_\Omega f(x)\ dx$ is finite.

Exercise 3. Consider the series

$$\sum_{n=1}^\infty \frac{1}{n} \sin\left(\frac{x}{n}\right)$$

.

(1) Does it converge uniformly on $[0, 1]$?
(2) Does it converge uniformly on $[0, \infty)$?

Exercise 4. Determine if

$$\sum_{n=1}^\infty \frac{\cos(k)}{k}$$

converges.

Exercise 5. For $a, b \geq 0$, let

$$F(a, b) = \int_{-\infty}^\infty \frac{dx}{x^4 + (x-a)^4 + (x-b)^4}.$$

For what values of $p \in (0, \infty)$ is

$$\int_0^1 \int_0^1 F(a, b)^p\ da\ db < \infty?$$

Hint: First, consider the case that $a \leq b$.

Exercise 6. Let $\sum_{n=1}^\infty a_n$ be a convergent series. Let $b_n \in \mathbb{R}$ be an increasing sequence with $\lim_{n\to\infty} b_n = \infty$. Show that

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^n b_k a_k = 0.$$