DAY 9 PROBLEMS AND SOLUTIONS

Exercise 1. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set with finite Lebesgue measure $m(E)$. Define $f(r) = m((E + r) \cap E)$. Prove that f is continuous.

Solution 1. Problems like this come up a lot. There is a trick to them and like every problem with a trick, they are easiest when you know the trick. The trick is to approximate the function χ_E in L^1 (or any L^p, $p \in [1, \infty)$) with continuous functions f_n, prove that $Tf_n : r \mapsto \int \chi_{E+r}(x) f_n(x)\ dx$ is continuous for all n, then use Young's inequality to prove that T is continuous from L^1 to $C(\mathbb{R})$, so it sends the convergent sequence $f_n \to \chi_E$ to a convergent sequence in $C(\mathbb{R})$, and hence $T\chi_E$ is continuous. If that is clear to you, you can stop reading here, but I'll fill in more details and point out a connection to a large collection of problems in analysis in the next couple paragraphs.

If you want to calculate the measure of $(E+r) \cap E$ (or any set you don't know the measure of), one thing to try is to write it as the integral of a characteristic function, because we are often more comfortable manipulating integrals than measures. This is especially powerful when we want to find the measure of an intersection, because $\chi_{A \cap B} = \chi_A \chi_B$. Applying this, we see that

$$f(r) = \int \chi_{E+r}(x)\chi_E(x)\ dx = \int \chi_E(x - r)\chi_E(x)\ dx.$$

We could write this as an integral operator $T_K f(r) = \int K(x, r) f(x)\ dx$, where $K(x, r) = \chi_E(x - r)$. This problem asks you to prove a special case of a general principle that integral operators of this form should improve the regularity of the input function. In this case, we want to prove that T_K sends a function in $L^2(\mathbb{R})$ (we could choose other values of $p \in (1, \infty)$ here) to a function in $C(\mathbb{R})$, and we have the very useful property that $K(x, r) = F(r - x)$ for a function $F \in L^2(\mathbb{R})$. We will approach the remainder of the problem in this framework.

With our choice of K and F, we have that $T_K f(r) = F * f(r)$, so by Young's inequality

$$||T_K f||_{L^\infty} \leq ||F||_{L^2} ||f||_{L^2}.$$

Therefore T_K boundedly maps $L^2(\mathbb{R})$ into $L^\infty(\mathbb{R})$. Now, let's check that T_K sends $C_0(\mathbb{R}) \to C(\mathbb{R})$. Suppose $f \in C_0(\mathbb{R})$ and let E denote its support. Fix $\varepsilon > 0$. Since f is continuous and compactly supported, it is uniformly continuous, so we can find small δ such that if $|r - s| < \delta$, then $|f(r) - f(s)| < \varepsilon / (||F||_{L^2} m(E)^{1/2})$. Applying this, we see that if $|r - s| < \delta$, by Hölder's inequality

$$|T_K f(r) - T_K f(s)| < \int |F(x)||f(r-x) - f(s-x)|\ dx < ||F||_{L^2} m(E)^{1/2} \sup_{x \in \mathbb{R}} |f(r-x) - f(s-x)| < \varepsilon$$

We are almost done now. We know that T_K sends $C_0(\mathbb{R})$ to $C(\mathbb{R})$ and is bounded $L^2(\mathbb{R}) \to L^\infty(\mathbb{R})$. Now we want to prove that it sends $L^2(\mathbb{R}) \to C(\mathbb{R})$. Recall that $C_0(\mathbb{R})$ is dense in L^2. We also see that $C(\mathbb{R})$ is closed in L^∞. Since the L^∞ norm is the sup norm for continuous functions, a sequence of functions f_n in $C(\mathbb{R})$ converging to f in the L^∞ norm is Cauchy in the sup-norm, and hence has a continuous limit. But since limits are unique, that limit must be f, so $f \in C(\mathbb{R})$. Now if we have $f \in L^2(\mathbb{R})$, then we can take a sequence $f_n \in C_0(\mathbb{R})$ with $f_n \to f$ in L^2. Then $T_K(f_n) \to T_K f$ in L^∞ and each $T_K(f_n)$ is continuous, so $T_K f$ is continuous as well. We can take $K(x,r) = \chi_E(x-r)$ and $f(x) = \chi_E(x)$ to solve the original problem.

Exercise 2. Let $I = [a,b]$ and let $L^2(I)$ be the space of square-integrable functions on $[a,b]$ with scalar product $\langle f, g\rangle = \int_a^b f(t)\overline{g}(t)\ dt$. Let $p_0, p_1, \dots$ be a sequence of real-valued polynomials p_n of degree exactly n such that

$$\int_a^b p_j(x)p_k(x)\ dx = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}.$$

Prove that $\{p_n\}_{n=0}^{\infty}$ is a *complete* orthonormal system.

Solution 2. A complete orthonormal system is an orthonormal system with dense span. It is immediate from the given definition that $\{p_n\}_{n\in\mathbb{N}}$ forms an orthonormal system, it suffices to show that it is complete. In other words, we need to prove that any $f \in L^2(I)$ can be approximated with polynomials. We have a very theorem about polynomial approximation called the Stone-Weierstrass theorem, which states that any continuous functions on $[a,b]$ can be approximated by polynomials in the $L^\infty([a,b])$ norm. The $L^2([a,b])$ norm is less than $\sqrt{b-a}$ times the $L^\infty([a,b])$ norm, so it follows that any continuous function on $[a,b]$ can be approximated in the $L^2([a,b])$ norm by polynomials as well. We then see that $\text{span}((p_n)_{n\in\mathbb{N}})$ is dense in $C([a,b])$ equipped with the $L^2([a,b])$ norm.

It remains to prove that $C([a,b])$ is dense in $L^2([a,b])$. This is a pretty standard result, but for completeness, I will prove this (or at least, reduce to very results I very much doubt you will be expected to prove). First, step functions are dense in $L^2([a,b])$, so it suffices to prove that $C([a,b])$ is dense in the set of $L^2([a,b])$ step functions. Since step functions are linear combinations of simple functions, it suffices to prove that $C([a,b])$ is dense the in the space of simple functions, in other words, that χ_E is a limit of continuous functions for any $E \in [a,b]$. By the regularity of the Lebesgue measure, for any $\varepsilon > 0$, we can find a compact set K and an open set U such that $K \subset E \subset U$ and $m(U \setminus K) < \varepsilon^2$. By Urysohn's lemma, we can find a continuous function $f : [a,b] \to [0,1]$ equal to 1 on K and 0 on U. Then $|f - \chi_E|$ is at most 1 and is only non-zero on $U \setminus K$. Thus, $||f - \chi_E||_{L^2([a,b])} < \varepsilon$. Since ε was arbitrary, we are done.

Exercise 3. Let X, Y be σ-finite measure space with positive measures $d\mu, d\nu$ respectively and let K be a measurable function on $X \times Y$. Let u and w be nonnegative measurable functions on X, Y respectively. Assume that $u(x) > 0$ and $w(y) > 0$ a.e.. Suppose $1 < p < \infty$

$$\int_X |K(x,y)|u(x)d\mu(x) \leq Bw(y), \quad \nu-\text{a.e.}$$

$$\int_Y |K(x,y)|w(y)^{1/(p-1)}d\nu(y) \leq Au(x)^{1/(p-1)} \quad \mu-\text{a.e.}$$

Let T be defined by $Tf(x) = \int_Y K(x,y)f(y)\ d\nu(y)$. Show that T maps $L^p(Y)$ to $L^p(X)$ with operator norm bounded by $A^{1-1/p}B^{1/p}$.

Solution 3. This is a tricky one and requires some functional analysis techniques that are not commonly discussed. Let's start with some simplifying assumptions: assume $X = Y = \mathbb{R}$, μ and ν are simply the Lebesgue measure, and $u \equiv w \equiv 1$. In hindsight, the first two assumptions will only provide moral support, as we will never use any properties unique to specific measure spaces or measures. The final assumption is more significant, but still will be easy enough to relax when the time comes. With these assumptions, are question is now to prove that if $\int_{\mathbb{R}} |K(x,y)|\ dx \leq B$ for almost every y and $\int_{\mathbb{R}} |K(x,y)|\ dy \leq A$ for almost every x, then if $Tf(x) = \int_{\mathbb{R}} K(x,y)f(y)\ dy$ satisfies $||Tf||_{L^p} \leq A^{1/p'}B^{1/p}||f||_{L^p}$, where p' is the Hölder conjugate of p. This feels more comfortable for me and hopefully for you as well.

A useful tool for proving L^p bounds of an integral operator is the principle of duality. For any $1 < p < \infty$, let p' be the Hölder conjugate of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$. Then

$$||Tf||_{L^p} = \sup_{g \neq 0 \in L^{p'}} \frac{\langle Tf, g\rangle}{||g||_{L^{p'}}} = \sup_{g \neq 0 \in L^{p'}} \frac{\int_{\mathbb{R}\times\mathbb{R}} K(x,y)f(y)g(x)\ dy\ dx}{||g||_{L^{p'}}}.$$

I think you are free to use this without proof, as it is somewhat challenging to prove. Doing this removes the strange asymmetry of having bounds in the integral of K over each individual variable, but an operator that only integrates over one variable. Now we see that it suffices to prove something much more symmetric:

$$\int_{\mathbb{R}\times\mathbb{R}} K(x,y)f(y)g(x)\ dy\ dx \leq A^{1/p'}B^{1/p}||f||_{L^p}||g||_{L^{p'}}.$$

It is easy to see that we only need to consider the case when K, f, and g are all non-negative, as the general case would follow from the fact that $\int_{\mathbb{R}\times\mathbb{R}} K(x,y)f(y)g(x)\ dy\ dx \leq \int_{\mathbb{R}\times\mathbb{R}} |K(x,y)||f(y)|g(x)|\ dy\ dx$.

Now, we will do something more unusual. Recall that Hölder's inequality holds for any measure space and non-negative measure, not just the Lebesgue measure. In particular, it holds for the measure $|K(x,y)|\ dx\ dy$. Applying this, we see that

$$\int_{\mathbb{R}\times\mathbb{R}} |K(x,y)||f(y)|g(x)|\ dy\ dx \leq \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |K(x,y)||f(y)|^p\ dx\ dy\right)^{1/p} \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |K(x,y)||g(y)|^{p'}\ dx\ dy\right)^{1/p'}$$

For the first integral, we will first integrate in x (since everything is non-negative, we are free to integrate in whatever order feels appropriate). Using the given bound $\int |K(x,y)|\ dx \leq B$, we see that the first integral is bounded above by $B^{1/p}||f||_{L^p}$. Similarly, the second integral is bounded above by $A^{1/p'}||g||_{L^{p'}}$. All together, $\int_{\mathbb{R}\times\mathbb{R}} |K(x,y)||f(y)|g(x)|\ dy\ dx \leq A^{1/p'}B^{1/p}||f||_{L^p}||g||_{L^{p'}}$, as desired.

Now, let's go back and remove the simplifying assumption. As stated previously, we need to change nothing to replace $\mathbb{R}$ with X and Y and dx, dy with $d\mu(x), d\nu(y)$. We do need to make some changes to account for u and w. But we start the same is the simplified case, applying duality to conclude that we need to prove $\int_{X\times Y} |K(x,y)||f(y)||g(x)|\ d\nu(y)\ d\mu(x) \leq A^{1/p'}B^{1/p}||f||_{L^p(d\nu)}||g||_{L^{p'}(d\mu)}$. Now, we need the u and w and appear somehow, so we will simply multiplying inside the integral by $1 = \frac{u^{1/p}(x)}{w^{1/p}(y)}\frac{w^{1/p}(y)}{u^{1/p}(x)}$. We now want to bound

$\int_{X\times Y} |K(x,y)||f(y)|\frac{u^{1/p}(x)}{w^{1/p}(y)}|g(x)|\frac{w^{1/p}(y)}{u^{1/p}(x)}\ d\nu(y)\ d\mu(x)$. Once again, we will use Hölder's inequality to see that

$$\int_{X\times Y} |K(x,y)||f(y)|\frac{u^{1/p}(x)}{w^{1/p}(y)}|g(x)|\frac{w^{1/p}(y)}{u^{1/p}(x)}\ d\nu(y)\ d\mu(x) \leq$$

$$\left(\int_{X\times Y} |K(x,y)||f(y)|^p\frac{u(x)}{w(y)}\ d\mu(x)\ d\nu(y)\right)^{1/p}\left(\int_{X\times Y} |K(x,y)||g(x)|^{p'}\frac{w^{p'/p}(y)}{u^{p'/p}(x)}\ d\mu(x)\ d\nu(y)\right)^{1/p'}$$

For the first integral we will, as previously, integrate in x first. The given bound tells us that

$$\left(\int_{X\times Y} |K(x,y)||f(y)|^p\frac{u(x)}{w(y)}\ d\mu(x)\ d\nu(y)\right)^{1/p} \leq \left(\int_Y \frac{B|f(y)|^p w(y)}{w(y)}\ dy\right)^{1/p} = B^{1/p}||f||_{L^p}.$$

Similarly (using the fact that $p'/p = \frac{1}{p-1}$), we see the second integral is bounded by $A^{1/p'}||g||_{L^{p'}}$. Putting this all together, we see that $\int_{X\times Y}|K(x,y)||f(y)||g(x)|\ d\nu(y)\ d\mu(x) \leq A^{1/p'}B^{1/p}||f||_{L^p(d\nu)}||g||_{L^{p'}(d\mu)}$, as desired.

Exercise 4. Suppose that $f \in L^1(\mathbb{R})$. Consider the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = \int_{\mathbb{R}} \frac{f(y)}{1+|xy|}\ dy.$$

(1) Prove that F is continuous,
(2) Prove that if $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, then $F \in L^p(\mathbb{R})$ for all $p > 2$.

Solution 4.

(1) We will use dominated convergence to prove that F is continuous. We want to prove $\lim_{x\to a} F(x) = F(a)$. Since $\lim_{x\to a} \frac{f(y)}{1+|xy|} = \frac{f(y)}{1+|ay|}$ and $\frac{|f(y)|}{1+|xy|}$ is uniformly bounded by the integrable function $|f(y)|$, the conditions for dominated convergence are satisfied and hence $\lim_{x\to a} F(x) = F(a)$. Therefore, F is continuous.

(2) Since we have $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, let's use Hölder's inequality on $\int_{\mathbb{R}} \frac{f(y)}{1+|xy|}\ dy$ in two different ways and see what comes out. First, let's put an L^1 norm on f and an L^∞ norm on $\frac{1}{1+|xy|}$. This gives that $|F(x)| \leq \int_{\mathbb{R}} \frac{|f(y)|}{1+|xy|}\ dy \leq ||f||_{L^1(\mathbb{R})}$, since $\sup_y \frac{1}{1+|xy|} = 1$. This bound never blows up, but it won't be helpful over all of $\mathbb{R}$, because it is constant. Now, let's put an L^2 norm on f and on $\frac{1}{1+|xy|}$. We need to calculate $\int_{\mathbb{R}} \frac{1}{(1+|xy|)^2}\ dy$, which might be a little tricky to do exactly, but if we do the substitution $u = |x|y$, then we realize that (so long as $x \neq 0$), $\int_{\mathbb{R}} \frac{1}{(1+|xy|)^2}\ dy = \frac{C}{|x|}$, where $C = \int_{\mathbb{R}} \frac{1}{(1+u)^2}\ du$. Then putting this for $||\frac{1}{1+|xy|}||^2_{L^2_y}$, we see that $|F(x)| \leq \frac{C||f||_{L^2}}{\sqrt{|x|}}$. This bound has decay in x, but blows up when x goes to 0.

So we have two bounds for F, one which constant in x and one which has decay in x but a singularity at 0. We can combine these to prove $\int_{\mathbb{R}} |F(x)|^p\ dx < \infty$ for $p > 2$ by using the constant bound in a neighborhood $[-1,1]$ of 0 and the decaying bound everywhere else. Specifically, since $p > 2$, $\int_{|x|>1} \frac{1}{|x|^{p/2}}\ dx < \infty$, so

$$\int_{\mathbb{R}} |F(x)|^p\ dx \leq \int_{-1}^{1} |F(x)|^p\ dx + \int_{|x|>1} |F(x)|^p\ dx \leq 2||f||^p_{L^1} + C||f||^p_{L^2}\int_{|x|>1}\frac{1}{|x|^{p/2}}\ dx < \infty.$$

Therefore, $F \in L^p$ for all $p > 2$.

Exercise 5. Let $E \subset [0,1]$ be a measurable set with positive Lebesgue measure. Morever, assume it satisfied the following property: as long as x and y belong to E, we know $\frac{x+y}{2}$ belongs to E. Prove that E is an interval.

Solution 5. Let $a = \inf E, b = \sup E$. Our aim will be to prove that $(a,b) \subset E$, which implies E is an interval.

First, let's prove that E contains a dense subset of (a,b). Let $a_n, b_n \in E$ be a sequence of elements with $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. We will prove that E contains a dense subset D_n of $[a_n, b_n]$ for any n. By taking the union over D_n, we arrive at a dense subset of (a,b). By inductively bisecting $[a_n, b_n]$ using the property that if $a, b \in E$, then $\frac{a+b}{2} \in E$, we see that

$$D_n = \bigcup_{m=1}^{\infty} \left\{ \frac{ja_n + (2^m - j)b_n}{2^m} : j = 0, 1, \ldots, 2^m \right\} \subset E$$

Now, let's prove that E contains an interval. This is a consequence of the standard result that the convolution of two functions in L^2 is continuous (see the solution to Exercise 1 for a careful proof of this). Since E has finite measure, $\chi_{E/2}$ is in L^2, so $\chi_{E/2} * \chi_{E/2}$ is continuous. We can check that $\chi_{E/2} * \chi_{E/2}$ is non-zero, because

$$\begin{aligned}
\int \chi_{E/2} * \chi_{E/2}(y)\, dy &= \int \int \chi_{E/2}(x)\chi_{E/2}(y-x)\, dx\, dy \\
&= \int \chi_{E/2}(x) \int \chi_{E/2}(y-x)\, dy\, dx \\
&= |E/2| \int \chi_{E/2}(x)\, dx \\
&= |E/2|^2 > 0.
\end{aligned}$$

Then we can find x where $\chi_{E/2} * \chi_{E/2}(x) > 0$, which, since $\chi_{E/2} * \chi_{E/2}$ is continuous, implies $\chi_{E/2} * \chi_{E/2}(y) > 0$ for y in some interval I containing x. If $\chi_{E/2} * \chi_{E/2}(y) > 0$, then there exists some z such that $\chi_{E/2}(z) > 0$ and $\chi_{E/2}(y-z) > 0$. It follows that $z, y-z \in \frac{E}{2}$, so since $\frac{E}{2} + \frac{E}{2} \subset E$, $y = z + (y-z) \in E$. Hence, $I \subset E$, as desired.

The final step is to prove that the longest interval in E has length (a,b). The geometric idea is if the longest interval I in E has length less than $b-a$, we can take some point c outside of I but very close to I, and look at $J = I \cup \frac{c+I}{2}$. Since $I \subset E$ and $c \in E$, $J \subset E$. If c is very close to I, then J will be an interval of length greater than I, contradicting the maximality of E. It follows that longest interval in E has length (a,b). To do this carefully, we need to carefully fill in the missing details.

Finally, let $m = \sup_{I\subset E} |I|$, where the supremum is over intervals I contained in E, and $|I|$ denotes the length of I. Since E contains an interval, this is well-defined. We will first prove that m achieves the supremum, that is, there exists an interval $I \subset E$ such that $m = |I|$. Otherwise, we have a sequence of intervals I_j with $\lim_{j\to\infty} |I_j| = m$. Write $I_j = (a_j, b_j)$. By compactness, we can restrict to a subsequence so that (as ordered pairs) $(a_{j_k}, b_{j_k}) \to (a', b')$, and since $\lim_{k\to\infty} b_{j_k} - a_{j_k} = m$, $b' - a' = m$. Then (as intervals), $(a', b') = \bigcup_{k\in\infty} I_{j_k} \subset E$, so the supremum is achieved by (a', b').

Now suppose $m < b - a$. Let's derive a contradiction. Let $I = (a', b') \subset E$ be an interval of length m. We know either $a < a'$ or $b > b'$, since otherwise $|I| = m \geq b - a$. Without loss of generality, assume $a < a'$. Since E contains a dense subset of (a,b), we can find $\varepsilon < \frac{m}{100}$ such that $c = a' - \varepsilon \in E$. Now consider $J = \frac{I+c}{2} \cup I$. We know that $a' + \frac{m}{100} \in I$ and

$a' + \frac{m}{100} = \frac{a' - \varepsilon + a' + \varepsilon + \frac{m}{50}}{2} \in \frac{I+c}{2}$, so since $\frac{I+c}{2}$ and I are interesecting intervals, J is an interval. We also now that $a' - \frac{\varepsilon}{4} = \frac{a' - \varepsilon + a' + \frac{\varepsilon}{2}}{2} \in \frac{I+c}{2}$, so $|J| \geq m + \frac{\varepsilon}{4}$, contradicting the maximality of I. Hence, $m = (b, a)$, so E is an interval.

Exercise 6. Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of Lebesgue measurable functions such that f_n converges to f a.e. on $[0,1]$ and such that $||f_n||_{L^2[0,1]} \leq 1$ for all n. Show that

$$\lim_{n\to\infty} ||f_n - f||_{L^1([0,1])} = 0.$$

Solution 6. Fix $\varepsilon > 0$. By Egerov's theorem, we can find a set A of measure $\leq \varepsilon$ where f_n converges uniformly on A^c. Then choosing n sufficiently large, we can ensure that $\sup_{x\in A} |f_n - f|(x) \leq \varepsilon$, so $\int_0^1 |f_n - f|(x)\ dx \leq \varepsilon$. On the other hand, by Hölder's inequality, $\int_{x\in A^c} |f_n - f|(x)\ dx \leq m(A^c)^{1/2} ||f_n - f||_{L^2} \leq \varepsilon^{1/2}(||f_n||_{L^2} + ||f||_{L^2})$. Let's check that $||f||_{L^2} \leq 1$. By Fatou's lemma and the pointwise a.e. convergence, $\int |f|^2(x)\ dx \leq \liminf_{n\in\mathbb{N}} \int |f_n|^2(x)\ dx \leq 1$, as desired. Then plugging this back in, we see that $\int_{x\in A^c} |f_n - f|(x)\ dx \leq 2\varepsilon^{1/2}$, and hence for n sufficiently large, $||f_n - f||_{L^1} \leq \varepsilon + 2\varepsilon^{1/2}$. Taking $\varepsilon \to 0$, we see that $\lim_{n\to\infty} ||f_n - f||_{L^1([0,1])} = 0$.