DAY 8 PROBLEMS AND SOLUTIONS

Exercise 1. Suppose $f_n \in L^1(\mathbb{R})$ and the sequence ε_n of positive real numers satisfies $\lim_{n\to\infty} \varepsilon_n = 0$. Define the sets $E_n = \{x : |f_n(x)| \geq \varepsilon_n\}$ and assume that $\sum_n m(E_n) < \infty$. Prove that

(1) $\lim_{n\to\infty} f_n = 0$ Lebesgue a.e. on $\mathbb{R}$.
(2) For every $\delta > 0$, there is a set Ω such that $m(\Omega) < \delta$ and $\lim_{n\to\infty} f_n = 0$ uniformly on $\mathbb{R} \setminus \Omega$.

Solution 1.

(1) By Borel-Cantelli, $\sum_n m(E_n) < \infty$ implies that for almost every x, $x \in E_n$ finitely often (in other words, $m(\limsup E_n) = 0$). Then for any x and for all n sufficiently large, $x \notin E_n$, and hence $|f_n(x)| < \varepsilon_n$. Since $\varepsilon_n \to 0$, $f_n(x) \to 0$ as well.
(2) Since $\sum_n m(E_n) < \infty$, we know that $\lim_{N\to\infty} \sum_{n=N}^{\infty} m(E_n) = 0$. Then for any δ, we can find N sufficiently large such that $\sum_{n=N}^{\infty} m(E_n) < \delta$. Set $\Omega = \bigcup_{n\geq N} E_n$. Then by subadditivity, $m(\Omega) < \delta$. For any $x \in \mathbb{R} \setminus \Omega$, we have $f_n(x) < \varepsilon_n$ for any $n \geq N$. Therefore, f_n converges uniformly to 0 on $\mathbb{R} \setminus \Omega$.

Exercise 2. Let $E \subset \mathbb{R}$. Suppose $g, f_n \in L^1(E)$, $\sup_n ||f_n||_{L^1} < \infty$ and $\lim_{n\to\infty} f_n = 0$ in Lebesgue measure. Prove that

$$\lim_{n\to\infty} \int_E \sqrt{|f_n g|}\, dx = 0.$$

Solution 2. We will use three different sources of "smallness" to control $\int_E \sqrt{|f_n g|(x)}\, dx$. One will be the convergence of f_n in measure. In other words, for any $\varepsilon > 0$ and any $\delta > 0$, for n sufficiently large, $|f_n|(x) > \varepsilon$ only occurs on a set of measure $< \delta$. The other two will come from the fixed function g.

I will prove both of the facts, although I would be surprised if you lost points for not doing so on the qual itself. Since $g \in L^1$, we know that for any $\varepsilon > 0$, we can find a finite measure set $F \subset E$ such that $\int_{E-F} |g|(x)\, dx < \varepsilon$. This is fairly easy to prove: if we define $F_x = E \cap [-x, x]$, then by the monotone convergence theorem, $\lim_{x\to\infty} \int_{F_x} |g|(y)\, dy = \int_E |g|(y)\, dy$, so we can find some x sufficiently large such that $\int_{E-F_x} |g|(y)\, dy < \varepsilon$. Our final fact is that for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $m(F) < \delta$, then $\int_F |g|(x)\, dx < \varepsilon$. Suppose this is not the case. Then there exists $\varepsilon > 0$ such that we can find sets $F_1, F_2, \ldots$ with $m(F_k) < 2^{-k}$ and $\int_E \chi_{F_k} |g|(x)\, dx > \varepsilon$. We know that $\lim_{k\to\infty} \chi_{F_k} = 0$ outside of a set of measure 0, so by dominated convergence, $\lim_{k\to\infty} \int_E \chi_{F_k} |g|(x)\, dx = 0$, contradicting $\int_E \chi_{F_k} |g|(x)\, dx > \varepsilon$ for all k.

With these three sources of smallness in hand, we will complete the proof. Let $M = \sup_m ||f_n||_{L^1}$. Fix $\varepsilon > 0$ and choose a set F such that $\int_F |g(x)|\, dx < \varepsilon^2/(9M)$ and $m(F^c) < \infty$. Choose $\delta > 0$ such that for any $G \subset E$ with $m(G) < \delta$, $\int_G |g(x)|\, dx < \varepsilon^2/(9M)$. Finally, for n sufficiently large, choose sets $H_n \subset F^c$ with $m(H_n) < \delta$ such that for all $x \in E - H_n$,

$|f_n(x)| < \varepsilon^2/(9m(F^c)||g||_{L^1})$. Write

$$\int_E \sqrt{|f_n g|(x)}\ dx = \int_F \sqrt{|f_n g|(x)}\ dx + \int_{H_n} \sqrt{|f_n g|(x)}\ dx + \int_{F^c - H_n} \sqrt{|f_n g|(x)}\ dx.$$

We will bound all three integrals using Cauchy-Schwarz. The first is bounded above by

$$\left(\int_F |f_n|(x)\ dx \int_F |g|(x)\ dx\right)^{1/2} < (||f_n||_{L^1}\varepsilon/(9M))^{1/2} < \varepsilon/3.$$

The second is bounded above by

$$\left(\int_{H_n} |f_n|(x)\ dx \int_{H_n} |g|(x)\ dx\right)^{1/2} < (||f_n||_{L^1}\varepsilon^2/(9M))^{1/2} < \varepsilon/3.$$

The third is bounded above by

$$\left(\int_{F^c-H_n} |f_n|(x)\ dx \int_{F^c-H_n} |g|(x)\ dx\right)^{1/2} < (m(F^c)\varepsilon^2/(9m(F^c)||g||_{L^1})||g||_{L^1})^{1/2} < \varepsilon/3.$$

Hence, $\int_E \sqrt{|f_n g|(x)}\ dx < \varepsilon$ for n sufficiently large. Since ε was arbitrary, we are done.

Exercise 3. Suppose that on a set E of finite measure, $f_n \to f$ in measure and $g_n \to g$ in measure and f is finite a.e.. Prove that $f_n g_n \to fg$ in measure on E.

Hint: Can you prove $f_n^2 \to f^2$ in measure?

Solution 3. Suppose we can prove the hint. Then $(f_n + g_n)^2 \to (f+g)^2$, $f_n^2 \to f^2$, and $g_n^2 \to g^2$ in measure. It follows that $f_n g_n = \frac{(f_n+g_n)^2 - f_n^2 - g_n^2}{2} \to \frac{(f+g)^2 - f^2 - g^2}{2} = fg$ in measure, as desired.

It remains to prove the hint. Fix $\varepsilon > 0$. We know that $m(\{x : |f_n - f|(x) > \varepsilon^{1/2}\}) \to 0$, so $m(\{x : |f_n - f|(x)^2 > \varepsilon\}) \to 0$. Since ε was arbitrary, we therefore know that $|f_n - f|^2 \to 0$ in measure. We can expand this to $f_n^2 + f^2 - 2f_n f \to 0$. Then if we can prove $f_n f \to f^2$, we would have that $f_n^2 = f_n^2 + f^2 - 2f_n f + 2f_n f - f^2 \to f^2$ in measure, since $f_n^2 + f^2 - 2f_n f \to 0$ in measure and $2f_n f - f^2 \to f^2$ in measure.

So let's prove that $f_n f \to f^2$ in measure. Fix $\varepsilon > 0$ and $\delta > 0$, we need to find $N \in \mathbb{N}$ such that if $n \geq N$, then $m(\{x : |f_n f - f^2| > \varepsilon\}) < \delta$. Since f is finite a.e., for some K sufficiently large, $|f|(x) \leq K$ for all x in a set A such that $m(E \setminus A) < \frac{\delta}{2}$. Now, choose N sufficiently large so that $m(\{x : |f_n - f| > \frac{\varepsilon}{K}\}) < \frac{\delta}{2}$ for all $n \geq N$. This is possible because $f_n \to f$ in measure. Then for $n \geq N$,

$$\begin{aligned} m(\{x : |f_n f - f^2| > \varepsilon\}) &< m(E \setminus A) + m(\{x \in A : |f_n f - f^2| > \varepsilon\}) \\ &\leq \frac{\delta}{2} + m(\{x \in A : |f_n - f| > \frac{\varepsilon}{|f|}\}) \\ &\leq \frac{\delta}{2} + m(\{x \in A : |f_n - f| > \frac{\varepsilon}{K}\}) \\ &< \delta. \end{aligned}$$

Since δ, ε was arbitrary, we know that $m(\{x : |f_n f - f^2|(x) > \varepsilon\}) \to 0$ for all $\varepsilon > 0$, so $f_n f \to f^2$ in measure, completing the proof.

Exercise 4. Suppose f, f_n are Lebesgue measureable functions on $[0,1]$ finite a.e.. Show that $f_n \to f$ in measure if and only if

$$\lim_{n\to\infty} \int_0^1 \frac{|f_n - f|(x)}{1 + |f_n - f|(x)}\, dx = 0.$$

Solution 4. Suppose $f_n \to f$ in measure. Let f_{n_k} be a subsequence of f_n. Then it also converges in measure to f and hence has a subsequence $f_{n_{k_j}}$ converging pointwise almost everywhere. Note that $\frac{|f_{n_{k_j}} - f|(x)}{1+|f_{n_{k_j}} - f|(x)} \in [0,1]$, so by dominated converge,ce, $\lim_{j\to\infty} \int_0^1 \frac{|f_{n_{k_j}} - f|(x)}{1+|f_{n_{k_j}} - f|(x)}\, dx = \int_0^1 \lim_{j\to\infty} \frac{|f_{n_{k_j}} - f|(x)}{1+|f_{n_{k_j}} - f|(x)}\, dx = 0$. Since every subsequence of $\{\int_0^1 \frac{|f_n - f|(x)}{1+|f_n-f|(x)}\, dx\}$ has a subsubsequence converging to 0, we see that $\lim_{n\to\infty} \int_0^1 \frac{|f_n-f|(x)}{1+|f_n-f|(x)}\, dx = 0$.

Now suppose $\lim_{n\to\infty} \int_0^1 \frac{|f_n-f|(x)}{1+|f_n-f|(x)}\, dx = 0$. Fix $\varepsilon > 0$ and let $E_n = \{x \in [0,1] : |f_n - f|(x) > \varepsilon\}$. We need to show that $m(E_n) \to 0$. Note that for $x \in E_n$, $\frac{|f_n-f|(x)}{1+|f_n-f|(x)} = 1 - \frac{1}{1+|f_n-f|(x)} \geq 1 - \frac{1}{1+\varepsilon} = \frac{\varepsilon}{1+\varepsilon}$. Then $\int_0^1 \frac{|f_n-f|(x)}{1+|f_n-f|(x)}\, dx \geq E_n \frac{\varepsilon}{1+\varepsilon}$, so $m(E_n) \leq \frac{1+\varepsilon}{\varepsilon} \int_0^1 \frac{|f_n-f|(x)}{1+|f_n-f|(x)}$, and so since $\lim_{n\to\infty} \int_0^1 \frac{|f_n-f|(x)}{1+|f_n-f|(x)} = 0$, $\lim_{n\to\infty} m(E_n) = 0$ as well. Since $\varepsilon > 0$ was arbitrary, it follows that $f_n \to f$ in measure.

Exercise 5. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable, real-valued functions on a measure space X such that $f_n \to f$ pointwise as $n \to \infty$, where $f : X \to \mathbb{R}$, and suppose that for some constant $M > 0$,

$$\int |f_n|\, d\mu \leq M \quad \text{ for all } n \in \mathbb{N}.$$

(1) Prove that

$$\int |f|\, d\mu \leq M.$$

(2) Give an example to show that we may have $\int |f_n|\, d\mu = M$ for every $n \in \mathbb{N}$, but $\int |f|\, d\mu < M$.

(3) Prove that

$$\lim_{n\to\infty} \int ||f_n| - |f| - |f_n - f||\, d\mu = 0.$$

Solution 5.

(1) By Fatou's lemma, $\int |f|\, d\mu \leq \liminf \int |f_n|\, d\mu \leq M$.

(2) Take $f_n(x) = \chi_{[n,n+M]}$. Then $f_n \to 0$ pointwise everywhere and $\int |f_n|(x)\, dx = M$ for all n, but $\int 0\, dx = 0 < M$.

(3) We will use dominated convergence. Since $f_n \to f$ pointwise, $|f_n| - |f| - |f_n - f| \to 0$ pointwise, so as long as the conditions for dominated convergence are satisfied, we will have $\lim_{n\to\infty} \int ||f_n| - |f| - |f_n - f||\, d\mu = 0$. We just need to find an integrable for $||f_n| - |f| - |f_n - f||$. We want that upper bound to be an integrable function, and given part (1), a reasonable guess is $|f|$ or some multiple of $|f|$. By rearranging the triangle inequality, we know $|f_n - f| \geq |f_n| - |f|$, so $||f_n| - |f| - |f_n - f|| = |f_n - f| - |f_n| + |f|$. By the same reasoning, $|f_n - f| - |f_n| \leq |f|$, so $|f_n - f| - |f_n| + |f| \leq 2|f|$. Hence $||f_n| - |f| - |f_n - f|| \leq 2|f|$, and by part (1), we know $\int 2|f|\, d\mu \leq 2M$, so the

conditions for the dominated convergence theorem hold. As discussed previously, this completes the problem.

Exercise 6. Let $f \in L^1(\mathbb{R})$ satisfy $\int_a^b f(x)\,dx = 0$ for any two rational numbers $a < b$. Does it follow that $f(x) = 0$ for almost every x?

Solution 6. It does follow that $f(x) = 0$ almost everywhere. Suppose otherwise. Without loss of generality, we may assume there exists a set E of measure $\varepsilon > 0$ on which $f > 1$. By the outer regularity of the Lebesgue measure, there exists a sequence of open sets $V_n \supset E$ such that $m(V_n) \to m(E)$. On $\mathbb{R}$, any open set is a countable collection of open intervals. By lengthening the mth interval in the collection by less than $\frac{1}{n2^m}$, we can ensure each of the open intervals has rational endpoints. Call the new open set U_n and note that $m(U_n) \leq m(V_n) + \frac{1}{n}$, and hence $m(U_n) \to m(E)$, and $U_n \supset V_n \supset E$ for all n. We then see that χ_{U_n} converges to χ_E in measure, and hence $\chi_{U_{n_k}}$ converges χ_E almost everywhere for some subsequence n_k. We know that $\int \chi_{U_{n_k}} f(x)\,dx = 0$, so by dominated convergence, $\int \chi_E f(x)\,dx = 0$ as well, contradicting our assumption that $|f| > 1$ on E.

Exercise 7. For a Lebesgue measurable subset E of $\mathbb{R}$, denote by χ_E the indicator function of E. Let $\{E_n : n \in \mathbb{N}\}$ be a family of Lebesgue measurable subsets of $\mathbb{R}$ with finite measure and let f be a measurable function such that

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f(x) - \chi_{E_n}|\,dx = 0.$$

Prove that f is almost everywhere equal to the indicator function of a measurable set.

You've seen this problem before, but I'd invite you to think about a very short proof using some facts from modes of convergence.

Solution 7. See the solution in the day 6 solution.