DAY 7 PROBLEMS AND SOLUTIONS

Exercise 1. Prove that there does not exist $f \in L^1(\mathbb{R})$ such that $g * f(x) = g(x)$ for any $g \in C_0(\mathbb{R})$ and $x \in \mathbb{R}$.

Solution 1. You can solve this easily using Fourier analysis and Riemann-Lebesgue theorem, but I will present a measure-theoretic proof. The equation $g * f = g$ has a solution if f is allowed to be a Radon measure $C_0^*(\mathbb{R})$. Let δ_0 the dirac mass at 0, so $\int_{-\infty}^{\infty} g(y-x) d\delta_0 x = g(y)$. If an L^1 function f also had that property, then $g * (f - \delta_0) = 0$ for any function $g \in C_0(\mathbb{R})$. It follows that $\langle g, f - \delta_0 \rangle = 0$ ($\langle \cdot, \cdot \rangle$ denotes the dual pairing) for any $g \in C_0(\mathbb{R})$. But then $f - \delta_0 = 0 \in C_0^*(\mathbb{R})$, which implies that $f = \delta_0$. However, $\delta_0 \notin L^1(\mathbb{R})$, since it is not absolutely continuous with respect to the Lebesgue measure, so this is a contradiction.

Exercise 2. For any $n \geq 1$, show that there exists closed sets $A, B \subset \mathbb{R}^n$ with $|A| = |B| = 0$, but $|A + B| > 0$ (as usual $A + B = \{a + b : a \in A, b \in B\}$).

Solution 2. First, let's do this in $\mathbb{R}$. Let C denote the standard Cantor set. Recall that we can characterize elements of C by real numbers $x \in [0, 1]$ with a ternary representation containing only 0s and 2s (ternary representations are not unique but we only require one representation to be of the desired form, for example $1 = 0.222 \cdots \in C$). Then we can characterize elements of $C/2 = \{x \in [0, 1] : 2x \in C\}$ as real numbers with ternary representation containing only 0s and 1s. It is well known that the Cantor set has measure 0, so $C/2$ has measure 0 as well. Let $A = B = C/2$, let's prove that $A + B \supset [0, 1]$. Take $y \in [0, 1]$ and let $0.y_1 y_2 \ldots$ be a ternary expansion of y, where $y_i \in \{0, 1, 2\}$. We will construct elements $a \in A, b \in B$ such that $a + b = y$. Let $a = 0.a_1 a_2 \ldots$, where

$$a_i = \begin{cases} 0 & y_i = 0 \\ 1 & y_i \neq 0 \end{cases}$$

and $b = 0.b_1 b_2 \ldots$ where

$$b_i = \begin{cases} 0 & y_i \neq 2 \\ 1 & y_i = 2 \end{cases}.$$

Then $a_i + b_i = y_i$ and $a_i, b_i \in \{0, 1\}$ for all i, so $a \in A$, $b \in B$, and $a + b = y$, as desired. It follows that $A + B = [0, 1]$. Since $|[0, 1]| = 1$, we know that $|A + B| \geq 1 > 0$.

Now, for $\mathbb{R}^n$. Let $A_1, B_1 \subset [0, 1]$ denote the sets constructed in the past paragraph. Let $A = A_1 \times [0, 1]^{n-1}, B = B_1 \times [0, 1]^{n-1}$. The Cartesian product of a measure zero set with any other set has measure zero, so $|A| = |B| = 0$. Now, let's prove that $A + B \supset [0, 1]^n$. Take $y = (y^1, \ldots, y^n) \in [0, 1]^n$. As previous proven, we can find $a^1 \in A_1, b^1 \in B_1$ such that $a^1 + b^1 = y^1$. Set $a = (a^1, 0, \ldots, 0) \in A$ and $b = (b^1, y^2, \ldots, y^n) \in B$. Then $a + b = y$, so $y \in A + B$. Hence, $|A + B| \geq 1 > 0$.

Exercise 3. Given $\alpha \geq 0$ the α-dimensional Hausdorff measure of a set $X \subset \mathbb{R}^n$ is

$$\mathcal{H}^\alpha(X) = \liminf_{r\to 0}\{\sum_{i=1}^{\infty} r_i^\alpha : X \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), r_i < r \text{ for all } i\}$$

and the Hausdorff dimension is $\dim_H(X) = \inf\{\alpha \geq 0 : \mathcal{H}^\alpha(X) = 0\}$.

Prove the following:

(1) If $X \subset \mathbb{R}^n$ and μ is a finite Borel measure on $\mathbb{R}^n$ such that $\mu(X) > 0$ and $\mu(B(x,r)) \leq r^\alpha$ for all open balls $B(x,r)$, then $\dim_H(X) \geq \alpha$.
(2) If $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is the unit circle, the $\dim_H(S^1) = 1$.

Solution 3.

(1) We need to show that $\mathcal{H}^\alpha(X) > 0$. Suppose otherwise. Then for any $\varepsilon > 0$, there exists a collection of balls $B(x_1, r_1), B(x_2, r_2), \dots$ covering X with $\sum_{i=1}^\infty r_i^\alpha < \varepsilon$. Then

$$\mu(X) \leq \mu\left(\bigcup_{i=1}^{\infty} B(x_i, r_i)\right) \leq \sum_{i=1}^{\infty} \mu(B(x_i, r_i)) \leq \sum_{i=1}^{\infty} r_i^\alpha < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\mu(X) = 0$, a contradiction. Hence, $\mathcal{H}^\alpha(X) > 0$, so $\dim_H(X) \geq \alpha$.

(2) First, let's check that $\dim_H(S^1) \leq 1$. To do so, we need to prove that $\mathcal{H}^s(S^1) = 0$ for any $s > \alpha$. For any $r > 0$, let $x_i = (\cos(\theta_i), \sin(\theta_i))$, where $\theta_i = \frac{2\pi i r}{100}$, for $i = 1, 2, \dots, \lfloor \frac{100}{2\pi r}, \rfloor$. Then for any $x \in S^1$, $x = (\cos(\theta), \sin(\theta))$ for some $\theta \in [0, 2\pi]$, so if θ_i is the closest point to θ, then $|x - x_i| \leq |\cos(\theta) - \cos(\theta_i)| + |\sin(\theta) - \sin(\theta_i)| \leq 2|\theta - \theta_i| \leq \frac{1}{100r}$. Hence, $x \in B(x_i, r)$, so $S^1 \subset \bigcup_{i=1}^{\lfloor \frac{100}{2\pi r} \rfloor} B(x_i, r_i)$. Moreover, $\sum_{i=1}^{\lfloor \frac{100}{2\pi r}} r_i^s \leq \frac{100}{2\pi r} r^s = \frac{100}{2\pi} r^{s-1}$. Since $s - 1 > 0$, $\lim_{r\to 0} \frac{100}{2\pi} r^{s-1} = 0$, and hence $\mathcal{H}^s(S^1) = 0$. Since $s > \alpha$ was arbitrary, we see that $\dim_H(S^1) \leq 1$.

Now let's prove that $\dim_H(S^1) \geq 1$. We will use the previous part and radial integration to accomplish this. Define $\mu(A) = \frac{1}{100} \int_{S^1} \chi_A(\cos(\theta), \sin(\theta))\, d\theta$. This is a well-defined measure, you can either check this from the definition or recall that μ is the pushforward of the Lebesgue measure on $\mathbb{S}^1$ under the map $\theta \mapsto (\sin(\theta), \cos(\theta))$. Also, $\mu(S^1) = 2\pi > 0$, so it remains to check the measure condition for balls. For a ball $B(x,r)$, if $B(x,r) \cap S^1 = \emptyset$, then $\mu(B(x,r)) = 0 < r$. If $B(x,r) \cap S^1 \neq \emptyset$, then $\mu(B(x,r))$ is $\frac{1}{100}$ times length of the circular arc $B(x,r) \cap S^1$. Let θ_1, θ_2 be the angles at the endpoints of arc and note that the length of the circular arc is $|\theta_1 - \theta_2|$. Then $\mu(B(x,r)) = \frac{|\theta_1 - \theta_2|}{100}$. On the other hand, $(\cos(\theta_1), \sin(\theta_1)), (\cos(\theta_2), \sin(\theta_2)) \in B(x,r)$ and $B(x,r)$ is a convex set, so it contains a line of length $\sqrt{|\cos(\theta_1) - \cos(\theta_2)|^2 + |\sin(\theta_1) - \sin(\theta_2)|^2} \geq \frac{|\theta_1 - \theta_2|}{2}$ (one of the approximations $|\sin(\theta_1) - \sin(\theta_2)| \geq \frac{|\theta_1 - \theta_2|}{2}$ and $|\cos(\theta_1) - \cos(\theta_2)| \geq \frac{|\theta_1 - \theta_2|}{2}$ must hold for any value of $\theta_1, \theta_2 \in [0, 2\pi)$). The ball then must have radius $> \frac{|\theta_1 - \theta_2|}{4}$, so

$$\mu(B(x,r)) = \frac{|\theta_1 - \theta_2|}{100} \leq \frac{|\theta_1 - \theta_2|}{4} < r.$$

Therefore, by the first part, $\dim_H(S^1) \geq 1$, and hence $\dim_H(S^1) = 1$.

Exercise 4. Let $z_1, z_2, \dots, z_n$ be points on the unit circle $\mathbf{T} = \{|z| = 1\}$ in the complex plane. Let $E \subset \mathbf{T}$ satisfy $m(E) > 2\pi(1 - \frac{1}{n})$. Prove that E can be rotated so that all the points z_k fall into the rotated set, i.e., that there exists $\alpha \in \mathbf{T}$ such that $\alpha z_k \in E$ for $k = 1, 2, \dots, n$.

Solution 4. Let $A_k = \{\alpha \in \mathbf{T} : \alpha z_k \in E\}$, we want to prove that $A_1 \cap \cdots \cap A_n \neq \emptyset$. Since $\alpha z_k \in E$ if and only if $\alpha \in z_k^{-1}E$, we see that $A_k = z_k^{-1}E$. Rotations are measure preserving, so $|A_k| = |E| > 2\pi(1 - \frac{1}{n})$. Then if $A_1 \cap \cdots \cap A_n = \emptyset$, we can take the set-difference of both sides from $\mathbf{T}$ to see that $(\mathbf{T} \setminus A_1) \cup \cdots \cup (\mathbf{T} \setminus A_n) = \mathbf{T}$. Each $\mathbf{T} \setminus A_i$ has measure $< \frac{2\pi}{n}$, so $|(\mathbf{T} \setminus A_1) \cup \cdots \cup (\mathbf{T} \setminus A_n)| < 2\pi = |\mathbf{T}|$, a contradiction. Therefore, $A_1 \cap \cdots \cap A_n \neq \emptyset$, so there exists $\alpha \in \mathbf{T}$ such that $\alpha z_k \in E$ for all k.

Exercise 5. Let σ be a Borel probability measure on $[0, 1]$ satisfying

(1) $\sigma([1/3, 2/3]) = 0$;
(2) $\sigma([a, b]) = \sigma([1 - b, 1 - a])$ for any $0 \leq a < b \leq 1$;
(3) $\sigma([3a, 3b]) = 2\sigma([a, b])$ for any a, b such that $0 \leq 3a < 3b \leq 1$.

Complete the following with justification:

(1) Find $\sigma([0, 1/8])$.
(2) Calculate the second moment of σ, i.e. the integral

$$\int_0^1 x^2 \, d\sigma(x).$$

Solution 5. I think of σ as the "uniform" measure supported on the Cantor set. That is not explicitly used in my solutions here, but might provide some context for how I came to the solution below.

(1) Since σ is a probability measure, it has total mass 1. The third property implies that $\sigma([0, 1/3]) = \frac{\sigma([0,1])}{2} = \frac{1}{2}$, $\sigma([0, 1/9]) = \frac{\sigma([0,1/3])}{2} = \frac{1}{4}$ and that $\sigma([1/9, 2/9]) = \frac{\sigma([1/3,2/3])}{2} = 0$. It follows that $\sigma([0, 1/8]) = \sigma([0, 1/9]) + \sigma([1/9, 1/8]) = \frac{1}{4}$.

(2) The given properties are listed in terms of the measure σ. For this problem, we need to turn the properties into properties of the integral $\int_0^1 f(x) \, d\sigma(x)$. The first property is immediate, but for the other two, you could formally check this by expressing the properties as $\sigma = T_*\sigma$ for appropriate choices of T, then recalling how to integrate pushforward measures. I don't think you would actually need to do this on the qual, so I will leave that as an exercise.

The properties are as follows:

(a) $\int_{1/3}^{2/3} f(x) \, d\sigma(x) = 0$,
(b) $\int_0^1 f(x) \, d\sigma(x) = \int_0^1 f(1 - x) \, dx$, and
(c) $\int_0^1 f(x) \, d\sigma(x) = 2 \int_0^{1/3} f(3x) \, d\sigma(x)$.

By the first property

$$\int_0^1 x^2 \, d\sigma(x) = \int_0^{1/3} x^2 \, d\sigma(x) + \int_{2/3}^1 x^2 \, d\sigma(x).$$

By the second property,

$$\int_{2/3}^1 x^2 \, d\sigma(x) = \int_0^{1/3} (1 - x)^2 \, d\sigma(x) = \int_0^{1/3} 1 \, d\sigma(x) + \int_0^{1/3} x^2 \, d\sigma(x) - 2 \int_0^{1/3} x \, d\sigma(x).$$

So $\int_0^1 x^2\, d\sigma(x) = 2\int_0^{1/3} x^2\, d\sigma(x) + \sigma([0,1/3]) - 2\int_0^{1/3} x\, d\sigma(x)$. By the third property, we see that $2\int_0^{1/3} x^2\, d\sigma(x) = \frac{2}{9}\int_0^{1/3}(3x)^2\, d\sigma(x) = \frac{1}{9}\int_0^1 x^2\, d\sigma(x)$. Substituting this in, we see that $\frac{8}{9}\int_0^1 x^2\, d\sigma(x) = \sigma([0,1/3]) - 2\int_0^{1/3} x\, d\sigma(x)$. In this way, we have reduced integrating x^2 to integrating x. We will proceed similarly to reduce integrating x to integrating 1. For brevity, I will not explain each step, but they all follow from applying the properties above.

We know that

$$
\begin{aligned}
2\int_0^{1/3} x\, d\sigma(x) &= \frac{2}{3}\int_0^1 (3x)\, d\sigma(x) \\
&= \frac{1}{3}\int_0^1 x\, d\sigma(x) \\
&= \frac{1}{3}\int_0^{1/3} x\, d\sigma(x) + \frac{1}{3}\int_{2/3}^{3} x\, d\sigma(x) \\
&= \frac{1}{3}\int_0^{1/3} x\, d\sigma(x) + \frac{1}{3}\int_0^{1/3} (1-x)\, d\sigma(x) \\
&= \frac{1}{3}\int_0^{1/3} 1\, d\sigma(x) \\
&= \frac{1}{3}\sigma([0,1/3]).
\end{aligned}
$$

Hence, $\frac{8}{9}\int_0^1 x^2\, d\sigma(x) = \sigma([0,1/3]) - \frac{1}{3}\sigma([0,1/3])$, and therefore $\int_0^1 x^2\, d\sigma(x) = \frac{3}{4}\sigma([0,1/3])$. Now $\sigma([0,1/3]) = \frac{1}{2}\sigma([0,1])$, by the third property, so since σ has mass one, $\sigma([0,1/3]) = \frac{1}{2}$. Therefore, $\int_0^1 x^2\, d\sigma(x) = \frac{9}{8}\cdot\frac{1}{3} = \frac{3}{8}$.

Exercise 6. Assume that for every $x \in (0,1)$, the function f is absolutely continuous on $[0,x]$ and bounded variation on $[x,1]$. Assume also that f is continuous at 1. Prove that f is absolutely continuous on $[0,1]$.

Solution 6. A function f is absolutely continuous if and only if it is BV, continuous, and maps measure zero sets to measure zero sets. The function f is clearly BV, it is continuous at 1 by assumption and at any $x < 1$ because it is absolutely continuous on $[0, \frac{1+x}{2}]$. If N is a measure zero set, then $|f(N)| = \lim_{x\to 1} |f(N\cap[0,x])| = 0$, since f is absolutely continuous on $[0,x]$, so it maps measure zero subsets of $[0,x]$ to measure zero sets and hence $f(N\cap[0,x]) = 0$ for all x.