DAY 6 PROBLEMS AND SOLUTIONS

Exercise 1. Let $f_n : X \to \overline{\mathbb{R}}$ be a sequence of measurable functions on a finite measure space X, so that $|f_n(x)| < \infty$ for almost every $x \in X$. Show that there is a sequence A_n of positive real numbers so that

$$\lim_{n\to\infty} \frac{f_n(x)}{A_n} = 0$$

almost everywhere. *Hint: Borel-Cantelli.*

Solution 1. The Borel-Cantelli lemma states that if E_n is a sequence of measurable sets and $\sum_{n\in\mathbb{N}} m(E_n) < \infty$, then $\limsup E_n$ has measure 0. It would be rather convenient if we could choose our sets E_n such that if $x \notin \limsup E_n$ (that is, $x \notin E_n$ for all but finitely many n), then $\lim_{n\to\infty} \frac{f_n(x)}{A_n} = 0$. We then could choose E_n to be the set of xs where $\frac{f_n(x)}{A_n}$ decays slower than a certain rate (say, $\frac{f_n(x)}{A_n} > 1/n$). We know that outside of $\limsup E_n$, $\frac{f_n(x)}{A_n} \leq 1/n$ for all n sufficiently large, and hence $\lim_{n\to\infty} \frac{f_n(x)}{A_n} = 0$. It then suffices to prove that $\sum_{n\in\mathbb{N}} m(E_n) < \infty$, and hence $\limsup E_n = 0$. We will give an explicit construction the E_n, then prove that $\sum_{n\in\mathbb{N}} m(E_n) < \infty$, completing the problem

Since f_n is finite a.e. and X has finite measure, $\lim_{c\to\infty} m(\{x : f_n(x) > c\}) = 0$. It follows that we can choose A_n sufficiently large so that $m(\{x : f_n(x) > A_n/n\}) < 2^{-n}$. Let $E_n = \{x : f_n(x) > A_n\}$. Since f_n is a measurable function, E_n is a measurable set. Additionally, $\sum_{n\in\mathbb{N}} m(E_n) < 1$. As discussed previously, Borel-Cantelli implies that $\limsup E_n$ has measure 0 and outside of $\limsup E_n$, $\lim_{n\to\infty} \frac{f_n(x)}{A_n} = 0$, completing the problem.

Exercise 2. Suppose $E \subset \mathbb{R}^d$ is a given set and O_n is the open set $O_n = \{x : d(x, E) < 1/n\}$.

(1) Show that if E is compact, then $|E| = \lim_{n\to\infty} |O_n|$.
(2) Is the statement false for E closed and unbounded?
(3) Is the statement false for E open and bounded?

Solution 2.

(1) We have that $|E| = \int_{\mathbb{R}^d} \chi_E(x)\, dx$ and $|O_n| = \int_{\mathbb{R}^n} \chi_{O_n}(x)\, dx$. Since E is compact, it is contained in $B_R(0)$ for R sufficiently large. It follows that $O_i \subset B_{R+1}(0)$, so $\int_{\mathbb{R}^d} \chi_{O_n}(x)\, dx < \infty$ for all n. Let's prove that $\lim_{n\to\infty} \chi_{O_n}(x) = \chi_E(x)$ for all x. If $x \in E$, then $\chi_{O_n}(x) = \chi_E(x) = 1$ for all x, and hence $\lim_{n\to\infty} \chi_{O_n}(x) = 1 = \chi_E(x)$. If $x \notin E$, then since E is compact, $d(x, E) = c > 0$. Now take N such that $\frac{1}{N} < c$. Then $x \notin O_n$ for any $n \geq N$, so $\chi_{O_n}(x) = 0$ for all $n \geq N$. It follows that $\lim_{n\to\infty} \chi_{O_n}(x) = 0 = \chi_E(x)$. Hence, $\lim_{n\to\infty} \chi_{O_n}(x) = \chi_E(x)$ for all x. Then by dominated convergence,

$$\lim_{n\to\infty} |O_n| = \lim_{n\to\infty} \int_{\mathbb{R}^d} \chi_{O_n}(x)\, dx = \int_{\mathbb{R}^d} \lim_{n\to\infty} \chi_{O_n}(x)\, dx = \int_{\mathbb{R}^d} \chi_E(x)\, dx = |E|.$$

(2) The statement is false. Consider $E = \mathbb{Z}$. This is discrete and hence closed, but $|O_n| = \infty$ for all n, while $|E| = 0$, so $\lim_{n\to\infty} |O_n| = \infty \neq |E|$.

(3) The statement is false. First, a counterexample in $\mathbb{R}$. Let C be the 1/4-Cantor set. Convince yourself that C has positive measure (unlike the standard Cantor set) as well as being closed and having has empty interior (like the standard Cantor set). Then $E = [0,1] \setminus C$ is open, dense in $[0,1]$, and has measure < 1. But since E is dense, $O_n = [0,1]$ for all n, and hence $\lim_{n\to\infty} |O_n| = 1 > |E|$ for all n.

A counterexample in higher dimensions is not strictly necessary, but taking the Cartesian product of the $\mathbb{R}$ counterexample with $(0,1)^{n-1}$ will give you one.

Exercise 3. Let $X = [0,1]$ with Lebesgue measure and $Y = [0,1]$ with counting measure. Give an example of a measurable function $f : X \times Y \to [0,\infty)$ for which Fubini's theorem does not apply. (This example shows that the theorem is not valid if the hypothesis of σ-finiteness is omitted.)

Solution 3. Let $f(x,y) = \begin{cases} 1 & x = y \\ 0 & \text{otherwise} \end{cases}$. Then $\int_X \int_Y f(x,y)\ dy\ dx = \int_X 1\ dx = 1$, while $\int_Y \int_X f(x,y)\ dx\ dy = \int_Y 0\ dy = 0$.

Exercise 4. For a Lebesgue measurable subset E of $\mathbb{R}$, denote by χ_E the indicator function of E. Let $\{E_n : n \in \mathbb{N}\}$ be a family of Lebesgue measurable subsets of $\mathbb{R}$ with finite measure and let f be a measurable function such that

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f(x) - \chi_{E_n}|\ dx = 0.$$

Prove that f is almost everywhere equal to the indicator function of a measurable set.

Solution 4. I think the most natural thing to do is to prove the sets $L_\varepsilon = \{x : ||f|(x) - 1| > \varepsilon \text{ and } ||f(x)| - 0| > \varepsilon\}$ have measure 0 by computing $\int_{L_\varepsilon} |f(x) - \chi_{E_n}(x)|\ dx$, then conclude that that f must equal 1 or 0 a.e.. I will give a less natural proof, which is quicker but requires a little more machinery.

Suppose χ_{E_n} converges pointwise a.e.. to f. The limit of a pointwise convergent sequence taking values in $\{0,1\}$ must either by 0 or 1, so f takes values in $\{0,1\}$ a.e.. Therefore, f a.e. equals $\chi_{f^{-1}(\{1\})}$. It is possible that χ_{E_n} does not converge a.e.., but since it converges in L^1, it has a subsequence E_{n_k} that converges a.e.. The subsequence still satisfies $\lim_{k\to\infty} \int |E_{n_k} - f|\ dx = 0$, so we follow the same proof to conclude that f is a characteristic function.

The problem asks for E to be measurable, which it is because f is measurable (since it is in L^1) and so $f^{-1}(\{1\})$ is measurable (technically, that set is only determined up to sets of measure zero, but sets of measure zero are always Lebesgue measureable).

Exercise 5. Let $K \subset \mathbb{R}^d$ be compact and let μ be a regular Borel measure on K with $\mu(K) = 1$. Prove that there exists a compact set $K_0 \subset K$ such that $\mu(K_0) = 1$ but $\mu(H) < 1$ for every compact $H \subsetneq K_0$.

Solution 5. Let $K_0 = \bigcap K'$, where the intersection is taken over all compact subsets K' of K with $\mu(K') = 1$. The intersection of compact sets is always compact, so K_0 is compact. Moreover, if $H \subsetneq K_0$ satisfies $\mu(H) = 1$, then H is in the family of compact sets with measure 1, and hence $K_0 \subset H$, a contradiction. Hence, $\mu(H) < 1$ for every compact $H \subset K_0$. It remains to prove that $\mu(K_0) = 1$. Clearly, $\mu(K_0) \leq 1$, so suppose $\mu(K_0) < 1$. Since μ is regular, $\mu(K_0) = \inf\{\mu(U) : K \subset U, U \text{ open}\}$. Then if $\mu(K_0) < 1$, there exists $U \subset K$ with $K_0 \subset U$ and $\mu(U) < 1$. Since $K_0 \subset U$, $K \setminus U \subset K \setminus K_0 = \bigcup U'$, where the union is taken over all open sets with measure 0. We can assume this union is countable by noting that $\bigcup U' = \bigcup V'$, where the union is taken over all basis elements with measure 0, and noting that there are only countably many basis elements. It follows that $\mu\left(\bigcup V'\right) \leq \sum \mu(V') = 0$, so since $\mu\left(K \setminus U\right) \leq \mu\left(K \setminus K_0\right)$, $\mu\left(K \setminus U\right) = 0$, so $\mu(U) = 1$, contradicting our earlier assumption that $\mu(U) < 1$.

AN: I do not believe the condition that μ is regular is necessary for this problem.