DAY 5 PROBLEMS AND SOLUTIONS

Exercise 1. Consider the space $C([0,10])$ of continuous functions on $[0,10]$, and for a given large number L consider the metric $d_L(f,g) = \max_{x\in[0,10]} e^{-Lx}|f(x) - g(x)|$.

(1) Argue that $C([0,10])$ with the metric d_L is a complete metric space.
(2) Show that there is a unique function which is continuous on $[0,10]$ and satisfies
$$f(x) = -15 + \cos(x) \int_0^x e^{e^{tx}} f(t)\, dt$$
for all $x \in [0,10]$.

Solution 1.

(1) Note that d_0 is the usual sup metric on $C([0,10])$, which is well-known to be complete. Also, $d_L(f,g) = d_0(e^{-Lx}f, e^{-Lx}g)$, so d_L must be a well-defined metric. Since $d_0(f,g)e^{-10L} \le d_L(f,g) \le d_0(f,g)$, d_L and d_0 have the same Cauchy sequences and convergent sequence. Since d_0 is complete, d_L must be complete as well.
(2) We need to choose an appropriate value of L so that we can apply the contraction mapping theorem. Let $Tf(x) = -15 + \cos(x) \int_0^x e^{e^{tx}} f(t)\, dt$. Then
$$d_L(Tf, Tg) \le \sup_{x\in[0,10]} \int_0^x e^{e^{tx} - L(x-t)} e^{-Lt} |f(t) - g(t)|\, dt.$$
Suppose we can prove that for any x and functions $f, g \in C([0,10])$,
$$\int_0^x e^{e^{tx} - L(x-t)} e^{-Lt} |f(t) - g(t)|\, dt < \sup_{t\in[0,10]} e^{-Lt}|f(t) - g(t)|,$$
or equivalently, that $\int_0^x e^{e^{tx} - L(x-t)}\, dt < 1$ for any $x \in [0,10]$. If we had this, then since $[0,10]$ is compact, $\sup_{x\in[0,10]} \int_0^x e^{e^{tx} - L(x-t)}\, dt := q < 1$, and hence $d_L(Tf, Tg) \le q d_L(f,g)$, and the contraction mapping theorem gives the desired result. We will then aim to prove that for L sufficiently large, $\int_0^x e^{e^{tx} - L(x-t)}\, dt < 1$.

Integrands can be small because the integrand is small or because the domain of integration is small. We will need to take advantage of both reasons to prove the desired bound, because if $x - t$ can be arbitrarily close to 0, then we have no hope of making $e^{tx} - L(x-t) < 0$, as e^{tx} will approach 1 while $L(x-t)$ will approach 0. Let ε_0 be a small constant, to be determined later. For $x > \varepsilon_0$, we can write $\int_0^x e^{e^{tx} - L(x-t)}\, dt = \int_0^{x-\varepsilon_0} e^{e^{tx} - L(x-t)}\, dt + \int_{x-\varepsilon_0}^x e^{e^{tx} - L(x-t)}\, dt$. The latter integral is bounded above by $\varepsilon_0 e^{e^{100}}$, so taking ε_0 sufficiently small, we can ensure that it is less than $1/2$. This will be our choice for ε_0. Now, we will choose L large enough to bound the first integral. We know that $x - t > \varepsilon_0$, so $L(x-t) > L\varepsilon_0$. On the other hand $e^{tx} \le e^{100}$. If we take L large enough, $e^{100} - L\varepsilon_0 < -\log(20)$, so $e^{tx} - L(x-t) < -\log(20)$, and hence $e^{e^{tx} - L(x-t)} < 1/20$. Then $\int_0^{x-\varepsilon_0} e^{e^{tx} - L(x-t)}\, dt < (x - \varepsilon_0)/20 < 1/2$. Thus, $\int_0^x e^{e^{tx} - L(x-t)}\, dt < 1$ if $x > \varepsilon_0$. If $x \le \varepsilon_0$, then $\int_0^x e^{e^{tx} - L(x-t)}\, dt \le \varepsilon_0 e^{e^{100}} < 1/2$. Either way, $\int_0^x e^{e^{tx} - L(x-t)}\, dt < 1$, so we are done.

Exercise 2. Prove that there are two functions $f_1, f_2 \in C[0,1]$ that solve the following system of equations for all $x \in [0,1]$,

$$20f_1(x) + 3f_2(x) = \sin(x) + \int_0^1 \sin(xt)\sin(f_1(t))\ dt$$

$$-f_1(x) + 10f_2(x) = \cos(x) - \int_0^{1/2} \cos(xt)\cos(f_2(t))\ dt.$$

Solution 2. This is a contraction mapping problem, which means we want to reformulate it into finding a fixed point of an operator $T : M \to M$ for some complete metric space M. Define $A = \begin{bmatrix} 20 & 3 \\ -1 & 10 \end{bmatrix}$ and

$$T_1 f(x) = \sin(x) + \int_0^1 \sin(xt)\sin(f_1(t))\ dt, \text{ and } T_2 f(x) = \cos(x) - \int_0^{1/2} \cos(xt)\cos(f(t))\ dt.$$

We want to solve $A[f_1, f_2]^T = [T_1 f_1, T_2 f_2]^T$. Since A is invertible (it's determinant is 203), this is equivalent to finding a fixed point of the operator $T(f_1, f_2) = A^{-1}[T_1 f_1, T_2 f_2]^T$, where $T : C_0(\mathbb{R})^2 \to C_0(\mathbb{R})^2$ and $C_0(\mathbb{R})^2$ is equppied with the sup-norm for each element: $d((f_1, f_2), (g_1, g_2)) = \sup_{x\in[0,1]} |f_1(x) - g_1(x)| + \sup_{x\in[0,1]} |f_2(x) - g_2(x)|$. To prove that $d(T(f_1, f_2), T(g_1, g_2)) < d((f_1, f_2), (g_1, g_2))$, first note that A^{-1} is itself a contraction. Since A is linear and invertible, it suffices to prove that $||Ax||^2 > c||x||^2$ for all $x \in \mathbb{R}^2 - \{0\}$ and some $c > 1$ (since $\mathbb{R}^2$ is finite dimensional, it would actually suffice to prove that $c \geq 1$). It's easy enough to see that A's least eigenvalue is $15 - \sqrt{22}$, so we can take $c = 15 - \sqrt{22} > 1$.

Now we will prove that $\tilde{T}(f_1, f_2) = [T_1 f_1, T_2 f_2]^T$ is a contraction. For any $f_1, f_2, g_1, g_2 \in C_0(\mathbb{R})$, we have

$$\begin{aligned} d(\tilde{T}(f_1, f_2),&\tilde{T}(g_1, g_2)) \\ &\leq \sup_{x\in[0,1]} \int_0^1 |\sin(xt)||\sin(f_1(t)) - \sin(g_1(t))|\ dt \\ &+ \sup_{x\in[0,1]} \int_0^{1/2} |\cos(xt)||\cos(f_2(t)) - \cos(g_2(t))|\ dt. \end{aligned}$$

Using the bounds $|\sin(\theta)|, |\cos(\theta)| < 1$ and $|\sin'(\theta)|, |\cos'(\theta)| < 1$, we conclude that

$$\sup_{x\in[0,1]} \int_0^1 |\sin(xt)||\sin(f_1(t)) - \sin(g_1(t))|\ dt < \sup_{x\in[0,1]} |f_1(x) - g_1(x)| \text{ and}$$

$$\sup_{x\in[0,1]} \int_0^{1/2} |\cos(xt)||\cos(f_2(t)) - \cos(g_2(t))|\ dt < \sup_{x\in[0,1]} |f_2(x) - g_2(x)|.$$

Thus, $\tilde{T}$ is a contraction. It follows that T is the composition of contractions and hence is a contraction as well, so by the contraction mapping theorem, it has a fixed point (f_1, f_2), which solves the given equations.

Exercise 3. Can one find a bounded sequence of real numbers $x_n, n \in \mathbb{Z}$ that satisfies

$$x_n = \sin(n) + 0.5x_{n-1} + 0.4\sin(x_{n+1})$$

for every $n \in \mathbb{Z}$?

Solution 3. This looks slightly different than other contraction mapping problems, but it is one. We need a metric to apply the contraction mapping theorem. The simplest metrics on sequence spaces are given by ℓ^p norms. In this case, our operator will map the sequence $x_n \equiv 0$ to the sequence $x_n = \sin(n)$, so it will not map into an ℓ^p space other than ℓ^∞. We will proceed using the ℓ^∞ metric.

Let $\ell^\infty(\mathbb{Z})$ be the space of integer indexed sequences of real numbers such that the norm $||(x_n)_{n\in\mathbb{N}}||_{\ell^\infty(\mathbb{Z})} := \sup_{n\in\mathbb{Z}} |x_n|$ is finite. Define the operator $T : \ell^\infty \to \ell^\infty$ by $T((x_n)_{n\in\mathbb{N}})_m = \sin(m) + 0.5x_{m-1} + 0.4\sin(x_{m+1})$. First, note that this is a well-defined mapping (that is, $T((x_n)_{n\in\mathbb{N}}) \in \ell^\infty(\mathbb{Z})$ for all $(x_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{Z})$), since

$$||T((x_n)_{n\in\mathbb{N}})||_{\ell^\infty(\mathbb{Z})} \leq 1.5 + 0.5||(x_n)_{n\in\mathbb{N}}||_{\ell^\infty(\mathbb{Z})} < \infty$$

To see that it is a contraction, note that

$$\begin{aligned} |(T((x_n)_{n\in\mathbb{N}}) - T((y_n)_{n\in\mathbb{N}}))_m| &\leq 0.5|x_{m-1} - y_{m-1}| + 0.4|\sin(x_{m+1}) - \sin(y_{m+1})| \\ &\leq 0.5|x_{m-1} - y_{m-1}| + 0.4|x_{m+1} - y_{m+1}| \\ &\leq 0.9||x - y||_{\ell^\infty(\mathbb{Z})}. \end{aligned}$$

Since m was arbitrary, we conclude that $||T((x_n)_{n\in\mathbb{N}}) - T((y_n)_{n\in\mathbb{N}})||_{\ell^\infty(\mathbb{Z})} \leq 0.9||(x_n)_{n\in\mathbb{N}} - (y_n)_{n\in\mathbb{N}}||_{\ell^\infty(\mathbb{Z})}$, so T is a contraction mapping. It follows that it has a fixed point $(x_n)_{n\in\mathbb{N}}$, which hence satisfies $x_n = \sin(n) + 0.5x_{n-1} + 0.4\sin(x_{n+1})$ for all n.

Exercise 4. Consider the following equation for an unknown function $f : [0,1] \to \mathbb{R}$:

$$f(x) = g(x) + \lambda \int_0^1 (x-y)^2 f(y)\ dy + \frac{1}{2}\sin(f(x)).$$

Prove that there exists a number λ_0 such that for all $\lambda \in [0, \lambda_0)$ and all continuous functions g on $[0,1]$, the equation has a continuous solution.

Solution 4. This is, of course, a contraction mapping problem. Let $T(f)(x) = g(x) + \lambda \int_0^1 (x-y)^2 f(y)\ dy + \frac{1}{2}\sin(f(x))$. Then

$$\begin{aligned} d(T(f_1), T(f_2)) &\leq \sup_{x\in[0,1]} \lambda \int_0^1 (x-y)^2 |f_1(y) - f_2(y)|\ dy + \frac{1}{2}|\sin(f_1(x)) - \sin(f_2(x))| \\ &\leq \sup_{x\in[0,1]} \lambda \int_0^1 (x-y)^2 |f_1(y) - f_2(y)|\ dy + \sup_{x\in[0,1]} \frac{1}{2}|\sin(f_1(x)) - \sin(f_2(x))|. \end{aligned}$$

Since $|\sin'(x)| \leq 1$ for all x, $\frac{1}{2}|\sin(f_1(x)) - \sin(f_2(x))| \leq \frac{1}{2}|f_1(x) - f_2(x)|$, so

$$\sup_{x\in[0,1]} \frac{1}{2}|\sin(f_1(x)) - \sin(f_2(x))| \leq \frac{1}{2}d(f_1, f_2).$$

Then it suffices to choose to chose λ small enough so that $\lambda \int_0^1 (x-y)^2 |f_1(y) - f_2(y)|\ dy \leq \frac{1}{3}d(f_1, f_2)$. To see that this can be done, note that $\int_0^1 (x-y)^2|f_1(y) - f_2(y)|\ dy \leq d(f_1, f_2)\int_0^1 (x-y)^2\ dy = d(f_1, f_2)\frac{x^3-(x-1)^3}{3}$. By the mean-value theorem, $\frac{x^3-(x-1)^3}{3} \leq \sup_{x\in[-1,1]} x^2 \leq 1$. Therefore, $\int_0^1 (x-y)^2|f_1(y) - f_2(y)|\ dy \leq d(f_1, f_2)$, so if we take $\lambda_0 = \frac{1}{3}$, then if $\lambda \leq \lambda_0$, then $\lambda \int_0^1 (x-y)^2 |f_1(y) - f_2(y)|\ dy \leq \frac{1}{3}d(f_1, f_2)$ and hence $d(T(f_1), T(f_2)) \leq \frac{5}{6}d(f_1, f_2)$. Then T is a contraction mapping, so it has a fixed point, which necessarily solves the given equation.

It is part of the contraction mapping theorem that such a fixed point is unique, but it is quite easy to prove as well. Suppose T has two fixed points f_1, f_2. Then since T is a contraction, $d(T(f_1), T(f_2)) < (1-\varepsilon)d(f_1, f_2)$ for some $\varepsilon \in (0,1)$, but $d(T(f_1), T(f_2)) = d(f_1, f_2)$, a contradiction unless both are zero. Hence, the fixed point is unique.

Exercise 5. Let K be a continuous function on $[0,1] \times [0,1]$ satisfying $|K| < 1$. Suppose that g is a continuous function on $[0,1]$. Show that there exists a continuous function f on $[0,1]$ such that

$$f(x) = g(x) + \int_0^1 f(y)K(x,y)\ dy.$$

Solution 5. This is a contraction mapping problem. Our operator will be $Tf(x) = g(x) + \int_0^1 f(y)K(x,y)\ dy$. Note that since K is continuous on a compact interval, it acheives it's supremum. Since $|K| < 1$, we therefore know $|K| \leq c$ for some $c < 1$. Then $|Tf_1(x) - Tf_2(x)| \leq \int_0^1 |f_1(y) - f_2(y)||K(x,y)|\ dy \leq c \sup_{x\in[0,1]} |f_1(x) - f_2(x)|$. It follows that $d(Tf_1, Tf_2) \leq cd(f_1, f_2)$, so T is a contraction mapping. Then by the contraction mapping theorem, it has a fixed point f. Hence, f satisfies $f(x) = g(x) + \int_0^1 f(y)K(x,y)\ dy$.

Exercise 6. Let $L : [0,1] \to [0,1]$ be a function satisfying

$$|L(x_2) - L(x_1)| \leq |x_2 - x_1|/4, |L(1/2) - 1/2| < 1/4.$$

Prove that there is a continuous function $f : [0,1] \to [0,1]$ satisfying

$$f(x) = (1-x)L(f(x)) + 1/100.$$

Solution 6. We are looking for a fixed point of a functional $Tf(x) = (1-x)L(f(x)) + \frac{1}{100}$ acting on the complete metric space M of continuous functions from $[0,1]$ to $[0,1]$, with sup metric. First, we need to check that is well-defined, that is, that $T(f) \in M$ for any $f \in M$. Since $|L(1/2) - 1/2| < 1/4$, we know that $L(1/2) \in (1/4, 3/4)$. Then for any $x \in [0,1]$, $|x - 1/2| < 1/2$, so $|L(x) - L(1/2)| \leq 1/8$, and hence $L(x) \in [1/8, 7/8]$. Then if $x \in [0,1]$, $(1-x)L(f(x)) \in [0, 7/8]$, so $(1-x)L(f(x)) + 1/100 \in [0,1]$. Hence, $Tf(x) \in [0,1]$ for all x, so $Tf \in M$, and $T : M \to M$ is a well-defined map.

Now, let's check that T is a contraction. For any $f, g \in C([0,1])$ and point x, $|Tf(x) - Tg(x)| \leq |1-x||L(f(x)) - L(g(x))| \leq |f(x) - g(x)|/4$. Hence, $\sup_{x\in[0,1]} |Tf(x) - Tg(x)| \leq \frac{1}{4} \sup_{x\in[0,1]} |f(x) - g(x)|$, so $d(Tf, Tg) \leq \frac{1}{4} d(f,g)$. It follows that T has a fixed point $f \in M$, as desired.