DAY 4 PROBLEMS AND SOLUTIONS

Exercise 1. Let $K : [a, b] \times [a, b] \to \mathbb{R}$ be a differentiable function such that

$$\max_{[a,b]^2} |K(x,t)| \leq 1, \max_{[a,b]^2} \left| \frac{\partial K}{\partial x}(x,t) \right| \leq 1.$$

Consider the space $C[a, b]$ of continuous functions on $[a, b]$ with the sup-norm. For $f \in C[a, b]$, define

$$Af(x) = \int_a^b K(x,t) f(t)\ dt.$$

(1) Prove that $\{Af : \max_{[a,b]} |f(x)| \leq 1\}$ is a totally bounded subset of $C[a, b]$.
(2) If in (1) we drop the assumption $\max_{[a,b]^2} \left|\frac{\partial K}{\partial x}(x,t)\right| \leq 1$ and keep the other assumptions, does $\{Af : \max_{[a,b]} |f(x)| \leq 1\}$ have to be a totally bounded subset of $C[a, b]$?

Solution 1.

(1) The usual definition of a totally bounded subset E of a metric space M is one where for any $\varepsilon > 0$, we can cover the set with finitely many ε-balls centered in E. This turns out to be equivalent to a set being precompact (that is, every sequence in E has a convergent subsequence), which will be a more convenient way to prove a set is totally bounded. We will prove the useful direction: suppose $E \subset M$ is not totally bounded. Then we can find an infinite sequence of ε-seperated points in M, which cannot be convergent, and hence M is not precompact. The contrapositive of what we have proved is that precompact sets must be totally bounded.

So it suffices to prove that $\{Af : \max_{[a,b]} |f(x)| \leq 1\}$ is precompact. Suppose $g_n = Af_n$ for some sequence f_n satisfying $\max_{x\in[a,b]} |f_n(x)| \leq 1$. We need to check two conditions to apply Arzela-Ascoli. First, we needed to check that $\{g_n\}$ is uniformly bounded. This is because $|Af_n(x)| \leq (b-a) \sup_{t\in[a,b]} |K(x,t)f(t)| \leq (b-a)$. Now, we need to prove that $\{g_n\}$ is uniformly equicontinuous. Fix $\varepsilon > 0$. For any $x, y \in [a, b]$ and any n,

$$|Af_n(x) - Af_n(y)| \leq \int_a^b |K(x,t) - K(y,t)| f(t)\ dt.$$

By the mean value theorem, $|K(x,t) - K(y,t)| \leq |x - y| \sup_{(\overline{x},t)\in[a,b]} \left|\frac{\partial K}{\partial x}(\overline{x},t)\right| \leq |x - y|$, so

$$|Af_n(x) - Af_n(y)| \leq (b-a)|x-y| \sup_{t\in[a,b]} |f(t)| \leq (b-a)|x-y|.$$

Hence g_n is uniformly equicontinuous. By Arzela-Ascoli, it has a convergent subsequence, so $\{Af : \max_{[a,b]} |f(x)| \leq 1\}$ is precompact, as desired.

(2) We will give essentially the same proof, but it will require a little more care without using the bound $\max_{[a,b]^2} \left|\frac{\partial K}{\partial x}(x,t)\right| \leq 1$. The only part that changes is proving equicontinuity. Since K is differentiable, it is continuous, and since $[a, b]$ is compact,

it is uniformly continuous. Fix $\varepsilon > 0$ and choose $\delta > 0$ such that if $||(x,t)-(y,s)|| < \delta$, then $|K(x,t) - K(y,s)| < \varepsilon/(b-a)$. Then

$$|Af_n(x) - Af_n(y)| < \int_a^b |K(x,t) - K(y,t)||f(t)|\ dt \le \varepsilon/(b-a) \int_a^b |f(t)|\ dt < \varepsilon.$$

We therefore have equicontinuity, and the rest of the proof follows.

Exercise 2. Let $K : [a,b] \times [a,b] \to \mathbb{R}$ be a continuous function. Consider the space $C[a,b]$ of continuous functions on $[a,b]$ with the sup-norm. For $f \in C[a,b]$, define

$$S_K f(x) = \int_a^b K(x,t) f(t)\ dt.$$

(1) Is $\{S_K f : \max_{[a,b]} |f(x)| \le 1\}$ necessarily a totally bounded subset of $C[a,b]$?
(2) Let f_n be a sequence of continuous functions on $[a,b]$ satisfying

$$\sup_n \sup_{x\in[a,b]} |f_n(x)| \le 1.$$

Does the sequence $S_K f_n$ necessarily have a convergent subsequence? Give a proof or counterexample.
(3) Let K_n be a sequence of continuous functions on $[a,b] \times [a,b]$ and assume that

$$\sup_n \max\{|K_n(x,y)| : (x,y) \in [a,b] \times [a,b]\} \le 1.$$

Let $f \in C[a,b]$. Does the sequence $S_{K_n} f$ necessarily have a convergent subsequence in $C[a,b]$? Prove or give a counterexample.

Solution 2.

(1) Yes. It suffices to prove any sequence in $E = \{S_K f : \max_{[a,b]} |f(x)| \le\}$ has a convergent subsequence, which will follow from the Arzela-Ascoli theorem. Take a sequence $S_K(f_1), S_K(f_2), \cdots \in E$. We need to prove this sequence is uniformly bounded and equicontinuous. Since K is continuous on a compact set, it is bounded above by some M. Then since each $\sup_{x\in[a,b]} |f_n(x)| \le 1$, $\sup_{x\in[a,b]} |S_K(f_n)|(x) \le (b-a)M$. To see equicontinuity, fix $\varepsilon > 0$. Since K is continuous on a compact interval, it is uniformly continuous, so we can find δ such that if $|x-y| < \delta$, then $|K(x,t) - K(y,t)| < \varepsilon/(b-a)$. We know that

$$|S_K(f_n)(x) - S_K(f_n)(y)| \le \int_a^b |K(x,t) - K(y,t)| f(t)\ dt < \varepsilon.$$

By Arzela-Ascoli, $S_K(f_n)$ has a convergent subsequence. It follows that $\{S_K f : \max_{[a,b]} |f(x)| \le 1\}$ must be totally bounded.
(2) Yes, as was proven in part (i).
(3) Without loss of generality, we may assume that $[a,b] = [-1/2, 1/2]$. Set $K_n(x,t) = \cos(nxt)$ and $f(x) = 1$. It is straightforward to compute that

$$S_{K_n} f(x) = \begin{cases} \frac{\sin(nx)}{nx} & x \ne 0 \\ 1 & x = 0 \end{cases}.$$

Suppose $\{S_{K_n} f(x)\}$ had a convergent subsequence. Then its limit f is continuous, but we have $f(0) = 1$ and $f(x) = 0$ for any $x \ne 0$, a contradiction. Hence, $\{S_{K_n} f\}$ cannot have a convergent subsequence.

Exercise 3. For $f \in L^2$, let $F(x) = \int_0^x f(t)\, dt$.

(1) Prove that
$$\int_0^1 \left(\frac{F(x)}{x}\right)^2\, dx \leq 4\int_0^1 f^2(x)\, dx.$$

(2) For $x \in [0,1]$
$$Af(x) = \frac{1}{x\sqrt{1+|\log(x)|}}\int_0^x f(t)\, dt.$$
Prove that if f_n is a sequence of continuous functions on $[0,1]$ with $\sup_n ||f_n||_{L^2([0,1])} \leq 1$, then Af_n has a subsequence converging in the $L^2([0,1])$ norm.

Solution 3.

(1) Using Cauchy-Schwartz, we see that $|F^2(x)| \leq x\int_0^x f^2(t)\, dt$. Then $\frac{|F^2(x)|}{x}\int_0^x f^2(t)\, dt$. By dominated convergence, $\lim_{x\to 0}\int_0^x f^2(t)\, dt = 0$, so $\lim_{x\to 0^+}\frac{|F^2(x)|}{x} = 0$. Using this and integrating by parts, we see that
$$\int_0^1 \left(\frac{F(x)}{x}\right)^2\, dx = \frac{F^2(x)}{x}\Big|_1^0 + 2\int_0^1 f(x)\frac{F(x)}{x}\, dx = -F^2(1) + 2\int_0^1 f(x)\frac{F(x)}{x}\, dx \leq 2\int_0^1 f(x)\frac{F(x)}{x}\, dx.$$
Then by Cauchy Schwartz, $\int_0^1 f(x)\frac{F(x)}{x}\, dx \leq \left(\int_0^1 f^2(x)\, dx\right)^{1/2}\left(\int_0^1 \frac{F(x)}{x}\, dx\right)^{1/2}$, so $\int_0^1\left(\frac{F(x)}{x}\right)^2\, dx \leq 2\left(\int_0^1 f^2(x)\, dx\right)^{1/2}\left(\int_0^1 \frac{F(x)}{x}\, dx\right)^{1/2}$. Rearranging and squaring, we arrive at the desired inequality.

(2) It would be great if we could apply Arzela-Ascoli directly to Af_n, but I don't think that is possible. But we can apply Arzela-Ascoli to $g_n(x) = \int_0^x f_n(t)\, dt$. To see that this is uniformly bounded, note that by Cauchy-Schwarz, $|g_n(x)| \leq x^{1/2}||f_n||_{L^2([0,1])} \leq 1$, since $x \in [0,1]$. We similarly have $|g_n(x) - g_n(y)| \leq |x-y|^{1/2}||f_n||_{L^2}$, so $\{g_n : n \in \mathbb{N}\}$ is equicontinuous. Then it has a uniformly convergent subsequence, g_{n_k} with limit g.

I first tried to prove that $Af_{n_k} \to \frac{g(x)}{x\sqrt{1+|\log(x)|}}$, but I didn't get anywhere doing that. I guessed we want to use part (a) somehow, which suggests that instead of comparing g_{n_k} to g (which might not be of the form $\int_0^x f(t)\, dt$), we should be comparing g_{n_k} to g_{n_j}. Conveniently, we know that g_{n_k} is a Cauchy sequence and, since L^2 is complete, it suffices to prove that Af_{n_k} is Cauchy.

Let's split up the integral to a part near 0 and everything else. Specifically,
$$||Af_{n_k} - Af_{n_j}||^2_{L^2([0,1])} = \int_0^\delta \frac{(g_{n_k}(x) - g_{n_j}(x))^2}{x^2(1+|\log(x)|)}\, dx + \int_\delta^1 \frac{(g_{n_k}(x) - g_{n_j}(x))}{x^2(1+|\log(x)|)}\, dx$$
By the previous part, $\int_0^\delta \frac{(g_{n_k}(x)-g_{n_j}(x))^2}{x^2(1+|\log(x)|)}\, dx \leq \frac{4}{1+|\log(\delta)|}\int_0^1 (f_{n_k}(x) - f_{n_j}(x))^2\, dx \leq \frac{C}{1+|\log(\delta)|}$ for some positive constant C, using the boundedness of $||f_n||_{L^2}$. On the other hand, $\int_\delta^1 \frac{(g_{n_k}(x)-g_{n_j}(x))^2}{x^2(1+|\log(x)|)}\, dx \leq \frac{1}{\delta^2}||g_{n_k} - g_{n_k}||^2_{L^\infty}$, using the fact that $\frac{1}{1+|\log(x)|} \leq 1$. Now to prove $||Af_{n_k} - Af_{n_j}||^2_{L^2([0,1])} < \varepsilon$, choose δ sufficiently small so that $\frac{1}{1+|\log(\delta)|} < \frac{\varepsilon}{2C}$ and then choose K sufficiently large so that for $k \geq K$, $||g_{n_k} - g_{n_k}||^2_{L^\infty} < \frac{\varepsilon\delta^2}{2}$. Combining these bounds, we see that for $k \geq K$, $||Af_{n_k} - Af_{n_j}||^2_{L^2([0,1])} \leq \varepsilon$. Since ε

was arbitrary, we conclude that Af_{n_k} is Cauchy in the L^2 norm and hence Af_n has a convergent subsequence.

Exercise 4. Suppose S is the set of real-valued functions continuous g on $[0,1]$ that satisfy two conditions:

$$\left|\int_0^1 g(x)\ dx\right| \leq 1$$

and

$$|g(x) - g(y)| \leq |x-y|^{1/2}$$

for each $x, y \in [0,1]$. Consider the functional

$$F(g) = \int_0^1 (1-5x^2) g^{10}(x)\ dx.$$

Is F bounded on S? Does it acheive it's maximum on S?

Solution 4. First, let's prove that F is bounded. The conditions on S do not individually imply that each g is bounded, but taken together, they imply that each $g \in S$ is bounded. Suppose that $g(x) > 2$ for some $x \in [0,1]$. Then since $|g(y) - g(x)| < 1$ for all $y \in [0,1]$, we have that $g(y) > 1$ for all $y \in [0,1]$, in which case $\int_0^1 g(y)\ dy > 1$, a contradiction. Similarly, we cannot have $g(x) < -2$. Then $|g(x)| < 2$ for all $x \in [0,1]$, so $|g^{10}(x)| < 2^{10}$ for all $x \in [0,1]$.

Now, apply Hölder's inequality to see that

$$|F(g)| \leq \int_0^1 (1-5x^2)\ dx \sup_{x\in[0,1]} |g^{10}(x)| < C$$

for some fixed constant C not depending on g.

To see that it achieves its maximum is somewhat trickier. Suppose $M = \sup_{g\in S} F(g)$. Then there is a sequence $g_n \in S$ such that $\lim_{n\to\infty} F(g_n) = M$. Let's prove that g_n subsequentially converges to some $g \in S$. Since F is bounded, it is continuous on S, and hence g will achieve the maximum.

We will prove g_n subsequentially converges using Arzela-Ascoli. We have already noted that elements of S are uniformly bounded. The second condition on elements of S also implies that S is equicontinuous. If we take $\varepsilon > 0$, then if $|x - y| < \varepsilon^2$, then for any $g \in S$, $|g(x) - g(y)| < \varepsilon$. It follows that S is an equicontinuous family, so g_n has a convergent subsequence g_{n_k}, converging to some g. We have already noted that the F is continuous, so $F(g) = M$. It remains to prove that $g \in S$. Since g_n is uniformly bounded, we can apply dominated convergence to see that $\lim_{k\to\infty} \int_0^1 g_{n_k}(x)\ dx = \int_0^1 g(x)\ dx$. Hence, $\left|\int_0^1 g(x)\ dx\right| \leq 1$. For other condition, we have that $|g(x) - g(y)| = \lim_{k\to\infty} |g_n(x) - g_n(y)| \leq \sqrt{|x-y|}$ for any pair x, y. Hence, $g \in S$ and F achieves its maximum on S.

Exercise 5. Suppose that $f_n : [0,1] \to \mathbb{R}$ is a sequence of continuous functions each of which has continuous first and second derivatives on $(0,1)$. Prove: If

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \text{for all} \quad x \in [0,1]$$

and

$$\sup_{n\geq 1} \max_{0<x<1} |f_n''(x)| < \infty,$$

then f' exists and is continuous on $(0, 1)$.

Solution 5. We will use Arzela-Ascoli on f'_n to prove that it has a uniformly convergent subsequence, then prove that if $f_n \to f$ uniformly and $f'_n \to g$ uniformly, then f is differentiable and $f' = g$.

Let $\sup_{n\geq 1} \max_{0<x<1} |f''_n(x)| = M$. Equicontinuity is easy: for any $x, y \in [0, 1]$, $|f'_n(x) - f'_n(y)| \leq |x - y| \max_{x<\xi<y} |f''_n(\xi)| \leq M|x - y|$, so f' is equicontinuous.

Uniform boundedness is a little tricker. If $f'_n(0)$ exists and is uniformly bounded, then $|f'_n(x) - f'_n(0)| \leq Mx \leq M$ for all $x \in [0, 1]$, by the mean value theorem, so as long as we can make sense of $f'_n(0)$ and prove it is bounded, we are good to go. We know for each n that $f'_n(x)$ forms a Cauchy sequence in x as $x \to 0$, since $|f'_n(x) - f'_n(y)| \leq M|x - y|$, so it must converge as $x \to 0$. Then by the mean value theorem, for any $x \in [0, 1]$, there exists $y \in (0, x)$ such that $\frac{f_n(x)-f_n(0)}{x} - f'_n(0) = f'_n(y) - f'_n(0)$, so since $f'_n(y) \to f'_n(0)$ as $x \to 0$, we have that $\lim_{x\to 0} \frac{f_n(x)-f_n(0)}{x} = f'_n(0)$, as desired. Now to prove that $f'_n(0)$ is uniformly bounded, note we can Taylor expand f_n at 0 to conclude that $f_n(1/2) = f_n(0) + \frac{f'_n(0)}{2} + c_n$, where c_n is bounded above by a constant times M (from the remainder form of the Taylor expansion, or the mean value theorem again). Then $f'_n(0) = 2(f_n(1/2) - f_n(0) - c_n)$. Since $f_n(0)$ and $f_n(1/2)$ are convergent sequences, they are bounded, $f'_n(0)$ is a bounded sequence, and hence $f'_n(x)$ is uniformly bounded.

Now we know that $f_{n_k} \to f$ uniformly and $f'_{n_k} \to g$ uniformly, so let's prove the claim I made in the first sentence, that f is differentiable and $f' = g$ (you could probably get away with stating this as a fact, but if you have time and can come up with the proof, it's worth including). I think the easiest way to do this is by proving that the integral of g coincides with $f(x)-f(0)$. Let $G(x) = \int_0^x g(t)\, dt$. Then since uniform limits on compact sets commute with integrals, $G(x) = \lim_{k\to\infty} \int_0^x f'_{n_k}(t)\, dt = \lim_{k\to\infty} f_{n_k}(x) - f_{n_k}(0) = f(x) - f(0)$. Then $f(x) = f(0) + \int_0^x g(t)\, dt$, so by the fundamental theorem of calculus, f is differentiable with derivative g.