DAY 3 PROBLEMS AND SOLUTIONS

Exercise 1. Take $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^n$ and let $X + Y = \{x + y : x \in X, y \in Y\}$.

(1) Assume X is closed and Y is compact. Prove that $X + Y$ is closed.
(2) If Y is closed but not compact, is $X + Y$ closed? Prove or give a counterexample.

Solution 1.

(1) Suppose z is a limit of elements of $X+Y$, that is, $\lim_{n\to\infty} x_n + y_n = z$ for $x_n \in X$ and $y_n \in Y$. Since Y is compact, y_n has a convergent subsequence y_{n_k}, with limit $y \in Y$. Then $z = \lim_{k\to\infty} x_{n_k} + y_{n_k} = \lim_{k\to\infty} x_{n_k} + y$. It follows that x_{n_k} must be convergent as well, and since X is closed, it's limit x must fall in X. Then $z = x + y \in X + Y$, so $X + Y$ is closed.
(2) No. Take $X = \{-n : n \in \mathbb{N}\}$ and $Y = \{n + 1/n : n \in \mathbb{N}\}$. Both are discrete and hence closed, but neither are compact. Then $1/n \in X + Y$ for all $n \in \mathbb{N}$, but $0 \notin X + Y$.

Exercise 2. A function $f : U \to \mathbb{R}$ defined on a subset $U \subset \mathbb{R}^n$ is

- *locally bounded* if for all $x \in U$ there exists $\varepsilon, R > 0$ such that $|f(y)| \le R$ for all $y \in U$ with $|x - y| < \varepsilon$,
- *globally bounded* if there exists $R > 0$ such that $|f(y)| \le R$ for all $y \in U$.

Prove: If $U \subset \mathbb{R}^n$, then the following are equivalent:

(1) U is compact,
(2) every locally bounded function $f : U \to \mathbb{R}$ is globally bounded.

Solution 2. Let's first prove (1) implies (2). Suppose U is compact and let f be a locally bounded function. For every $x \in U$, we have a ball $B(x, \varepsilon)$ on which f is bounded by R. Let $\mathcal{U}$ be the collection of such balls. This forms an open cover of U, so since U is compact, it has a finite subcover $\mathcal{U}' = \{B(x_1, \varepsilon_1), \ldots, B(x_m, \varepsilon_m)\}$, where f is bounded by R_i on $B(x_i, \varepsilon_i)$. Then f is bounded on all of U by $\max\{R_1, \ldots, R_m\}$, so f is globally bounded.

Now, let's prove (2) implies (1). We will prove U is closed and bounded. First, let's prove it is closed. Take y in the closure of U and suppose $y \notin U$. Then $f(x) = \frac{1}{|x-y|}$ is locally bounded: for any $x \in U$, let $r = |x - y|$. Then if $|x - z| < r/2$, $|x - y| \le |x - z| + |z - y|$, so $r/2 \le |x - y| - |x - z| \le |z - y|$. Hence, $\frac{2}{r} \ge \frac{1}{|z-y|} = f(z)$, so f is locally bounded. On the other hand, there exists a sequence $x_n \in U$ converging to y. Then $f(x_n) = \frac{1}{|x_n-y|}$ is unbounded, so f is not globally bounded, a contradiction. It follows that U must be closed.

To see that U is bounded, let $f(x) : U \to \mathbb{R}$ be given by $f(x) = |x|$. This is locally bounded, since if $|x - y| \le 1$, then $|f(y)| \le |x - y| + |x| \le |x| + 1$. Then f must be globall bounded by some R, so $U \subset B(0, R)$. Hence, U is bounded and closed, so it must be compact.

Exercise 3. Let $\mathcal{K}$ denote the collection of compact subsets of $[0,1]$. Define the Hausdorff metric on $\mathcal{K}$ by

$$d(K_1, K_2) = \sup_{x \in K_1} \inf_{y \in K_2} |x-y| + \sup_{x \in K_2} \inf_{y \in K_1} |x-y|.$$

Prove that $(\mathcal{K}, d)$ is a complete metric space.

Solution 3. First, let's check that d is a metric. It is hopefully clear that $d(K,K) = 0$ for any $K \in \mathcal{K}$. If K_1, K_2 are distinct compact sets, then, without loss of generality, there exists $x \in K_1 \setminus K_2$, in which case $d(K_1, K_2) > \inf_{y \in K_2} |y - x| > 0$, so $d(K_1, K_2) = 0$ if and only if $K_1 = K_2$. Since the definition of d is symmetric in it's inputs, $d(K_1, K_2) = d(K_2, K_1)$. Now take $K_1, K_2, K_3 \in \mathcal{K}$. We can bound by the triangle inequality

$$\begin{aligned}
\sup_{x \in K_1} \inf_{y \in K_2} |x-y| &\leq \sup_{x \in K_1} \inf_{y \in K_3} \inf_{z \in K_3} |x-z| + |z-y| \\
&\leq \sup_{x \in K_1} \inf_{z \in K_3} |x-z| + \inf_{y \in K_3} \inf_{z \in K_2} |z-y| \\
&\leq \sup_{x \in K_1} \inf_{z \in K_3} |x-z| + \sup_{z \in K_3} \inf_{y \in K_2} |z-y|.
\end{aligned}$$

By the same reasoning, $\sup_{y \in K_2} \inf_{x \in K_1} |x-y| \leq \sup_{y \in K_2} \inf_{z \in K_3} |z-y| + \sup_{z \in K_3} \inf_{x \in K_1} |x-z|$. Summing these two gives that $d(K_1, K_2) \leq d(K_1, K_3) + d(K_2, K_3)$. Thus, the triangle inequality holds.

Proving that the space is complete is quite a bit trickier. If you know what the lim sup and lim inf of a collection of sets are, you should hope that those will coincide with the limit in the Hausdorff topology, so this gives you an outline for how to approach this problem: prove that if K_n is a Cauchy sequence in $\mathcal{K}$, then it converges to $\limsup K_n$ (I'm guessing $\liminf K_n$ would work as well, but I'll leave that as an exercise for the reader). We actually want to be a little more careful than just taking $\limsup K_n$, we will take our putative limit to be $K = \bigcap_{n \in \mathbb{N}} \overline{\bigcup_{j \geq n} K_j}$. Taking the closure is necessary to ensure K is compact. For example, if $K_n = [0, 1 - 1/n]$, then $\limsup K_n = [0,1)$, while the actual limit should be $[0,1]$.

Now, let's prove $K_n \to K$ in the Hausdorff metric. Fix $\varepsilon > 0$ and choose N sufficiently large so that for any $n, m \geq N$, $d(K_n, K_m) < \varepsilon/100$. Take arbitrary $x \in K$, we will prove that $\inf_{y \in K_n} |x - y| < \varepsilon/2$ for $n \geq N$. Since $x \in \overline{\bigcup_{j \geq N} K_j}$, we can find some $m \geq N$ such that for some $y \in K_m$, $|x - y| < \varepsilon/100$. For any $n \geq N$, since $d(K_n, K_m) < \varepsilon/100$, we know $\inf_{z \in K_n} |y - z| < \varepsilon/100$, so there exists $z \in K_n$ such that $|y - z| < \varepsilon/100$. It follows that $|x - z| < |x - y| + |y - z| < \varepsilon/50$. Therefore, $\inf_{x \in K_n} |x - z| < \varepsilon/50$ for any $z \in K$, and hence $\sup_{z \in K} \inf_{x \in K_n} |x - z| < \varepsilon/50$.

Now take $n \geq N$ and $x \in K_n$. Since $d(K_n, K_m) < \varepsilon/100$ for any $m \geq N$, we know that for any $m \geq N$, there exists $y_m \in \overline{B}(x_n, \varepsilon/100)$. The sequence y_m is contained in a compact set, so it has a subsequential limit $y_{m_j} \to z \in \overline{B}(x_n, \varepsilon/100)$. Let's prove that $z \in K$. We need to prove that for any $k \in \mathbb{N}$ and any $\delta > 0$, there exists $a \in \bigcup_{m \geq k} K_m$ such that $|a - z| < \delta$. But we know that for j large enough, $y_{m_j} \in \bigcup_{m \geq k} K_m$ and $|y_{m_j} - z| < \delta$. Hence, $z \in \bigcap_{n \in \mathbb{N}} \overline{\bigcup_{j \geq n} K_j} = K$. Since $z \in \overline{B}(x_n, \varepsilon/100)$, $|z - x| \leq \varepsilon/100$. Therefore, $\inf_{z \in K} |x - z| \leq \varepsilon/100$. Since $x \in K_n$ was arbitrary, we see that $\sup_{x \in K_n} \inf_{z \in K} |x - z| \leq \varepsilon/100$. Finally, we conclude that $d(K_n, K) = \sup_{z \in K} \inf_{x \in K_n} |x - z| + \sup_{x \in K_n} \inf_{z \in K} |x - z| < \varepsilon$, as desired.

Exercise 4. Prove that any open set $U \subset \mathbb{R}^n$ can be expressed as a countable union of rectangles.

Solution 4. Let $\mathcal{U} = \{\prod_{i=1}^n (a_i, b_i) \subset U : a_i, b_i \in \mathbb{Q}\}$. Since $\mathbb{Q}$ is countable, $\mathcal{U}$ is countable. For any point $x \in U$, there exists $r > 0$ such that $B(x, r) \subset U$. We can find $q \in \mathbb{Q}$ with $|q - x| < r/100$ and $r_0 \in \mathbb{Q}$ with $r_0 \in [0, r/10]$. Then $R = \prod_{i=1}^n (q_i - r_0, q_i + r_0) \in \mathcal{U}$, $q \in R$, and $R \subset B(x, r) \subset U$. Then $U = \bigcup_{R \in \mathcal{U}} R$ and $\mathcal{U}$ is countable, as desired.

Exercise 5. Let $x_1, \dots, x_{n+1}$ be pairwise distinct real numbers. Prove that there exists $C > 0$ such that: if $P : \mathbb{R} \to \mathbb{R}$ is a polynomial with degree at most n, then

$$\max_{x \in [0,1]} |P(x)| \leq C \max\{|P(x_1)|, \dots, |P(x_{n+1})|\}.$$

Solution 5. This follows almost immediately from the fact that all norms on a finite dimensional vector space are equivalent, but I think that might be too powerful of a result for you to be allowed to use. I will give a direct proof instead. The open mapping theorem would also let me skip some steps, but is certainly not necessary for this proof.

Let $\mathcal{P}_n = \{P : \mathbb{R} \to \mathbb{R} \text{ a polynomial} : \deg(P) \leq n\}$ and equip this space with the sup-norm $||\cdot||_{L^\infty}$. Define the linear map $L : \mathcal{P}_n \to \mathbb{R}^{n+1}$ by $L(P) = (P(x_1), \dots, P(x_{n+1}))$. This is an injective function, since a degree n polynomial that vanishes at $n + 1$ points must be 0. Since $\dim(\mathbb{R}^{n+1}) = \dim(\mathcal{P}_n) = n + 1$, L is a bijection. Therefore, it has an inverse $L^{-1} : \mathbb{R}^{n+1} \to \mathcal{P}_n$ (it's not too hard to construct this explicitly, but probably longer to do so than to use argument I gave for it's existence). Linear maps are necessarily continuous, so $||P||_{L^\infty} = ||L^{-1}(L(P))||_{L^\infty} \leq C|L(P)|$, where $|\cdot|$ denotes the standard norm on $\mathbb{R}^{n+1}$. But $|L(P)| = \sqrt{P(x_1)^2 + \cdots + P(x_{n+1})^2} \leq (n + 1) \max\{|P(x_1)|, \dots, |P(x_{n+1})|\}$. Hence, $||P||_{L^\infty} \leq (n + 1)C \max\{|P(x_1)|, \dots, |P(x_{n+1})|\}$, as desired.

Exercise 6. Let $f \in C^1([0, 1])$. Show that for every $\varepsilon > 0$ there exists a polynomial p such that

$$||f - p||_\infty + ||f' - p'||_\infty < \varepsilon.$$

Solution 6. First, note that if $|f' - p'|(x) < \varepsilon/2$ for all x and if $f(0) = p(0)$, then $|f(x) - p(x)| \leq \int_0^x |f'(t) - p'(t)| \, dt \leq \varepsilon/2$. By Stone-Weierstrass, we can find a polynomial q which satisfies $||q - f'||_{L^\infty} < \varepsilon/2$. Then $p(x) = \int_0^x q(t) \, dt + f(0)$ is a polynomial, satisfies $p(0) = f(0)$, and $||p' - f'||_{L^\infty} < \varepsilon/2$ and therefore satisfies $||f - p||_{L^\infty} < \varepsilon/2$, solving the problem.

Exercise 7. Let $a = (a_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers. Prove that the set

$$X = \{(x_n)_{n \in \mathbb{N}} \in \ell^1(\mathbb{N}) : x_n \in [0, a_n] \text{ for all } n \in \mathbb{N}\}$$

is compact in the $\ell^1(\mathbb{N})$ norm if and only if $(a_n)_{n \in \mathbb{N}} \in \ell^1(\mathbb{N})$.

Solution 7. Let's start with the easy direction. Suppose X is compact. Define $a^m \in X$ for $m \in \mathbb{N}$ by $a^m_n = \delta_{n \leq m} a_n$. Then a^m must subsequentially converge to $b \in X \subset \ell^1(\mathbb{N})$, but $b_n = a_n$ for all $n \in \mathbb{N}$ (since $a_n^{m_j} = a_n$ if m_j is our subsequence from compactness and j is sufficiently large), so $b = a$, and hence $a \in \ell^1(\mathbb{N})$.

Now, for the hard direction. When I was taking the quals, I didn't know what a totally bounded set was, so I would have proven this by proving any sequence in X has a convergent subsequence. I am not going to present that argument here, but I invite you to give it a try. If you get stuck, you could consult the proof that totally bounded and closed sets are compact.

But now I know what a totally bounded set is, so I will prove it using that. Assume $a \in \ell^1(\mathbb{N})$. First, note that X is closed. If $a^m \in X$ converges to a limit b, then $a^m_n \to b_n$

for each $n \in \mathbb{N}$. Since each $a_n^m \in [0, a_n]$, it follows that $b_n \in [0, a_n]$ and hence $b \in X$. Now, let's prove X is totally bounded. Fix $\varepsilon > 0$. Since $a \in \ell^1$, there exists $N \in \mathbb{N}$ such that $\sum_{n>N} a_n < \frac{\varepsilon}{2}$. For each $n \leq N$, define a finite collection of reals $b_n^1, \dots b_n^{j_n} \in [0, a_n]$ such that the $\frac{\varepsilon}{2N}$ balls around the elements b_n^i cover $[0, a_n]$. Define the finite collection elements

$$B = \{(b_1^{j_{i_1}}, \dots, b_N^{j_{i_N}}, a_{N+1}, a_{N+2}, \dots) : 1 \leq j_{i_n} \leq j_n \text{ for } 1 \leq n \leq N\}.$$

Let U denote the collection of ε-balls centered at points in B. For any $x \in X$, $\sum_{n>N} |x_n - a_n| < \frac{\varepsilon}{2}$. For each $n < N$, we can find an element $b_n^{i_n}$ within $\frac{\varepsilon}{2N}$ of x_n. Then $b = (b_1^{i_1}, \dots, b_N^{i_N}, a_{N+1}, a_{N+2}, \dots) \in B$ and $||x - b||_{\ell^1} < N\frac{\varepsilon}{2N} + \frac{\varepsilon}{2} \leq \varepsilon$. Therefore, U covers X. Since this can be done for any $\varepsilon > 0$, we know that X is totally bounded, and because closed, totally bounded sets are compact, we are done.