DAY 2 PROBLEMS AND SOLUTIONS

Exercise 1. Let $\{a_n\}$ be a convergent sequence of complex numbers and let $\lim_{n\to\infty} a_n = L$. Let

$$c_n := \frac{1}{n^5}\sum_{k=1}^{n} k^4 a_k.$$

Prove that c_n converges and determine its limit.

Solution 1. By comparing with the integral x^4, we see that $\sum_{k=1}^{n} k^4$ is between $(n+1)^5/5$ and $n^5/5$, so a reasonable guess for the limit would be $L/5$. Let's see if we can prove this directly. We see that

$$|c_n - L/5| = \frac{1}{n^5}\sum_{k=1}^{n} |k^4 a_k - k^4 L| + \frac{1}{n^5}|n^5 L/5 - \sum_{k=1}^{n} k^4 L|. \tag{1}$$

The second sum is more easily bounded: the integral comparison test tells us it must be between 0 and $L\frac{n^5-(n+1)^5}{5n^5}$. Since $n^5 - (n+1)^5 = O(n^4)$ by the binomial expansion, the second integral is $O(1/n)$ and hence goes to 0.

For the first sum, fix $\varepsilon > 0$ and assume $|a_k - L| < \varepsilon/2$ for all $k \geq N$. Then for $n \geq N$, the first sum breaks up into

$$\frac{1}{n^5}\sum_{k=1}^{N} |k^4 a_k - k^4 L| + \frac{1}{n^5}\sum_{k=N+1}^{n} k^4 |a_k - L|. \tag{2}$$

Taking n sufficiently large, the first sum in (2) is $< \varepsilon/3$. Using the integral bound again, the second sum in (2) is $< \varepsilon/2 \frac{1}{n^5}\sum_{k=N+1}^{n} k^4 < \varepsilon/2\frac{(n+1)^5}{n^5}$. For n sufficiently large, $\frac{(n+1)^5}{n^5} < 3/2$, so the second sum is less than $2\varepsilon/3$. Hence, both sums on the right side of (1) go to 0 as n goes to ∞, so the limit converges to the desired bound.

Exercise 2. Does the improper integral

$$\int_2^\infty \frac{x\sin(e^x)}{x+\sin(e^x)}\, dx$$

converge?

Solution 2. This is a tricky problem. I'll present one solution.

First, let's prove that $\sin(e^x)$ is integrable. I later googled it and found out you can prove this by integrating by parts, but the sublevel set approach we discussed in class works as well (and I already typed it up). Let's estimate the integral over individual half-periods of $\sin(e^x)$. Let $a_k = \log(\pi 2k)$ and $b_k = \log(\pi(2k+1))$ (we can safely ignore the contribution from 2 to a_1). Between a_k and b_k, the integrand is positive and between b_k and a_{k+1}, the integrand is negative. We can analyze $\int_{a_k}^{b_k} \sin(e^x)\, dx$ by u-substitution. Let $u = e^x$, so $\frac{du}{u} = dx$. In this variable $\int_{\pi 2k}^{\pi(2k+1)} \frac{\sin(u)}{u}\, du$. Since sin is always positive in this interval, we see that this is between $\frac{1}{\pi 2k}\int_{\pi 2k}^{\pi(2k+1)} \sin(u)\ du = \frac{1}{k\pi}$ and $\frac{1}{\pi(2k+1)}\int_{\pi 2k}^{\pi(2k+1)} \sin(u)\ du = \frac{1}{\pi(k+1/2)}$. Putting this all together, we see that $\int_{a_k}^{b_k} \sin(e^x)\ dx \in \left(\frac{c}{k+1/2}, \frac{c}{k}\right)$ for $c = 1/\pi$.

We can proceed similarly to conclude that $\int_{b_k}^{a_{k+1}} \sin(e^x)\ dx \in \left(-\frac{c}{k+1/2}, -\frac{c}{k+1}\right)$. It follows that

$$\int_{a_k}^{a_{k+1}} \sin(e^x)\ dx \in \left(0, c\left(\frac{1}{k} - \frac{1}{k+1}\right)\right).$$

By the mean value theorem, $\frac{1}{k} - \frac{1}{k+1} \leq C\frac{1}{k^2}$ for some fixed constant C. Therefore

$$\int_{a_k}^{a_{k+1}} \sin(e^x)\ dx \in \left(0, \frac{cC}{k^2}\right).$$

We then see that for $y \in [a_k, a_{k+1}]$,

$$\int_{a_1}^{y} \sin(e^x)\ dx = \sum_{j=1}^{k-1}\int_{a_j}^{a_{j+1}} \sin(e^x)\ dx + \int_{a_k}^{y} \sin(e^x)\ dx \leq cC\sum_{j=1}^{k-1} j^{-2} + \int_{a_k}^{y} \sin(e^x)\ dx.$$

We know that $cC\sum_{j=1}^{k-1} j^{-2}$ converges in k. The estimates we have previously done imply that $\int_{a_k}^{y} \sin(e^x)\ dx \in \left(0, \frac{cC}{k}\right)$, which also converges in k. Since k is an increasing function of y, $\lim_{y\to\infty}\int_{a_1}^{y} \sin(e^x)\ dx$ converges in y as well. Hence, the given integral does converge.

Of course, that is not the integral we were asked to compute. But now we know that $\int_2^\infty \frac{x\sin(e^x)}{x+\sin(e^x)}\ dx$ converges if and only if $\int_2^\infty \frac{x\sin(e^x)}{x+\sin(e^x)} - \sin(e^x)\ dx$ converges, or equivalently, if and only if $\int_2^\infty \frac{\sin^2(e^x)}{x+\sin(e^x)}\ dx$ converges. This is better, because the integrand is strictly non-negative, so if we make the denominator of the integrand bigger, it makes the integral smaller. If $x > 2$, then $\sin(e^x) + x < 2x$, so

$$\int_2^\infty \frac{\sin^2(e^x)}{x+\sin(e^x)}\ dx > \frac{1}{2}\int_2^\infty \frac{\sin^2(e^x)}{x}\ dx.$$

We will prove the final integral diverges, completing the problem. Let's substitute $u = e^x$. The integral becomes $\int_{e^2}^\infty \frac{\sin^2(u)}{u\log(u)}\ du$. At this point. In a small neighborhood $I = (\pi/2 - \varepsilon, \pi/2)$, $\sin^2(u) \geq \frac{1}{2}$. Since $\sin^2(u)$ is 2π-periodic, $\sin^2(u) \geq \frac{1}{2}$ on $I_n = 2\pi n + I$ as well. Then if $a_n = 2\pi n + \pi/2$, we have

$$\int_{e^2}^\infty \frac{\sin^2(u)}{u\log(u)}\ du \geq \sum_{n=0}^\infty \int_{I_n} \frac{\sin^2(u)}{u\log(u)}\ du \geq \sum_{n=0}^\infty \frac{1}{a_n\log(a_n)}.$$

We can prove that $\sum_{n=0}^\infty \frac{1}{a_n\log(a_n)}$ diverges by the integral comparison test: since $\frac{1}{a_n\log(a_n)} \geq \frac{1}{(2\pi x+\pi/2)\log(2\pi x+\pi/2)}$ for $x \geq n$, we know that $\sum_{n=0}^\infty \frac{1}{a_n\log(a_n)} \geq \int_1^\infty \frac{1}{(2\pi x+\pi/2)\log(2\pi x+\pi/2)}\ dx$. By substituting $y = 2\pi x$ and then $z = y + \pi/2$, we see that this converges if and only if $\int_2^\infty \frac{1}{x\log(x)}\ dx$ converges. Let $u = \log(x)$, so $du = \frac{dx}{x}$, and the integral becomes the definitely divergent $\int_{\log(2)}^\infty \frac{1}{u}\ du$.

Exercise 3. Let f be a C^1 function on $[0, \infty)$. Suppose that

$$\int_0^\infty t|f'(t)|^2\ dt < \infty,$$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(t)\ dt = L.$$

Show that $f(t) \to L$ as $t \to \infty$.

Solution 3. This is kind of like a Cesaro summation problem, except with integrals, which I find makes things easier. Fix $\varepsilon > 0$, let's prove that for S sufficiently large and $t > S$, $|f(t) - L| < \varepsilon$. Since $\int_0^\infty t|f'(t)|^2\ dt < \infty$, for S_0 sufficiently large, $\int_{S_0}^\infty t|f'(t)|^2\ dt < (\varepsilon/3)^2$. Integrating by parts, we see that $\int_{S_0}^T f(t)\ dt = Tf(T) - S_0 f(S_0) - \int_{S_0}^T tf'(t)\ dt$. Therefore, $f(T) = \frac{\int_{S_0}^T f(t)\ dt}{T} + \frac{S_0 f(S_0)}{T} + \frac{\int_{S_0}^T tf'(t)\ dt}{T}$. The last term is the hardest to control, so we will do so first. By Cauchy-Schwartz (applied with respect to the measure $d\mu = t\ dt$), for $T \geq S_0$, we have

$$\int_{S_0}^T f'(t)t\ dt \leq \left(\int_{S_0}^T |f'(t)|^2 t\ dt\right)^{1/2} \left(\int_{S_0}^T t\ dt\right)^{1/2} \leq \frac{\varepsilon}{3}((T^2 - S_0^2)/2)^{1/2} \leq \frac{\varepsilon T}{3}.$$

Then $\frac{\int_{S_0}^T f'(t)t\ dt}{T} \leq \frac{\varepsilon}{3}$.

Clearly, taking T sufficiently large, we can ensure that $\left|\frac{S_0 f(S_0)}{T}\right| \leq \frac{\varepsilon}{3}$. Finally, we want to prove that for T sufficiently large, $\left|\frac{\int_{S_0}^T f(t)\ dt}{T} - L\right| < \frac{\varepsilon}{3}$. Since $\lim_{T\to\infty} \frac{\int_0^T f(t)\ dt - \int_{S_0}^T f(t)\ dt}{T} = \lim_{T\to\infty} \frac{\int_0^{S_0} f(t)\ dt}{T} = 0$, we know that $\lim_{T\to\infty} \frac{\int_{S_0}^T f(t)\ dt}{T} = \lim_{T\to\infty} \frac{\int_0^T f(t)\ dt}{T} = L$, imply that we can find T large enough so that $\left|\frac{\int_{S_0}^T f(t)\ dt}{T} - L\right| < \frac{\varepsilon}{3}$ holds.

Taking all the bounds we have estabilished, we see that $|f(T) - L| \leq \varepsilon$ for T sufficiently large. Since $\varepsilon < 0$ was arbitrary, we see that $\lim_{T\to\infty} f(T) = L$, as desired.

Exercise 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a compactly supported function that satisfies the Hölder condition with exponent $\beta \in (0, 1)$, that is, there exists a constant $A < \infty$ such that for all $x, y \in \mathbb{R}$, $|f(x) - f(y)| \leq A|x - y|^\beta$. Consider the function g defined by

$$g(x) = \int_{-\infty}^{\infty} \frac{f(y)}{|x - y|^\alpha}\ dy$$

where $\alpha \in (0, \beta)$

(1) Prove that g is a continuous function at 0.
(2) Prove that g is differentiable at 0. (Hint: Try the dominated convergence theorem.)

Solution 4.

(1) By the substitution $u = x - y$, $g(x) = \int_{-\infty}^\infty \frac{f(u-x)}{|u|^\alpha}\ du$. Assume f is supported on $[-M, M]$. Then if $|x| < 1$, and hence $f(u), f(u - y)$ are both supported on $u \in [-M - 1, M + 1]$, so

$$|g(0) - g(x)| \leq \int_{-M-1}^{M+1} \frac{f(u - x) - f(u)}{|u|^\alpha}\ dy < A|x|^\beta \int_{-M-1}^{M+1} \frac{1}{|u|^\alpha}\ du.$$

Since $\alpha < 1$, the integral of $|u|^{-\alpha}$ over $[-M - 1, M + 1]$ is some finite constant C, so $|g(0) - g(x)| \leq AC|x|^\beta$. Hence, g is continuous at 0.

(2) This is an example of what is called a kernel operator. On $\mathbb{R}$, these are operators T associated with a measurable function $K(x,y)$ defined to be $Tf(x) = \int K(x,y)f(y)\,dy$ (in this case, we take $K(x,y) = |x-y|^{-\alpha}$). These operators come up in many guises. A key property of these operators is that they tend to improve the regularity of the input function. Of course, this depends on the properties of K, but in general we should get more regularity in the output than we might expect in the input. In this problem, our kernel has a singularity at 0 and our function isn't differentiable, but the kernel operator outputs a differentiable function. There are various ways we can prove that increase in regularity. When you can figure out the derivative of your kernel, using integration by parts and putting the derivative on the kernel can be an effective way to prove the desired regularity. Using the substitution in the previous problem, we see that

$$\frac{g(x)-g(0)}{x} = \int_{-\infty}^{\infty} \frac{f(u-x)-f(u)}{x|u|^\alpha}\,du.$$

Integrating by parts, we send $u^{-\alpha}$ to $Cu^{-\alpha-1}$ for some constant C, and we send $f(u-x)-f(u)$ to $F(x,y) = \int_0^u f(y-x)-f(y)\,dy$. The boundary terms in the integration by parts vanish, so we see that

$$\frac{g(x)-g(0)}{x} = C\int_{-\infty}^{\infty} \frac{F(x,u)}{x|u|^{\alpha+1}}\,du$$

We can rearrange $F(x,u) = \int_0^{-x} f(y+u)-f(y)\,dy$, so $|F(u)| \leq A|x||u|^\beta$. Then $\left|\frac{F(x,u)}{x|u|^{\alpha+1}}\right| \leq A\frac{|x||u|^\beta}{|x||u|^{\alpha+1}} = \frac{A}{|u|^{\alpha+1-\beta}}$, which is integrable since $\beta > \alpha$. Then we are justified in using the dominated convergence theorem. By the fundamental theorem of calculus, $\lim_{x\to 0}\frac{F(x,u)}{x} = f(u)-f(0)$, so

$$\lim_{x\to 0}\frac{g(x)-g(0)}{x} = \int_{-\infty}^{\infty}\lim_{x\to 0}\frac{F(x,u)}{u^{\alpha+1}}\,du = \int_{-M}^{M}\frac{f(u)-f(0)}{|u|^{\alpha+1}}\,du,$$

where once again, $[-M,M]$ is the support of f. This final integral is finite because $|f(u)-f(0)| < |u|^\beta$, so $\lim_{x\to 0}\frac{g(x)-g(0)}{x}$ converges, and we are done.

Exercise 5. Show that $\int_0^\infty \frac{\sin(x)}{x^{2/3}}\,dx$ converges. Determine if

$$\int_1^\infty \frac{\sin(x)}{x^{2/3}+\sin(x)}\,dx$$

converges. *Hint: Use Taylor expansion.*

Solution 5. Let's first show that $\int_0^\infty \frac{\sin(x)}{x^{2/3}}\,dx$ converges. There are two ways this integral could diverge: it could grow to quickly at 0 or shrink to slowly at ∞. The first case is easy to rule out: $\sin(x) \leq x$, so $\int_0^1 \frac{\sin(x)}{x^{2/3}}\,dx \leq \int_0^1 x^{1/3}\,dx < \infty$. The second is not much harder to rule out. We will integrate by parts to conclude $\int_1^\infty \frac{\sin(x)}{x^{2/3}}\,dx = \cos(1) - \frac{2}{3}\int_1^\infty \frac{\cos(x)}{x^{5/3}}\,dx$. We can bound the absolute value of the final integral by $\int_1^\infty \frac{1}{x^{5/3}}\,dx$, which is finite. It follows that $\int_1^\infty \frac{\sin(x)}{x^{2/3}}\,dx$ converges as well.

The second integral converges as well. Once gain, we only need worry about divergence at 0 or at ∞. When $x \in [0,1]$, $\sin(x) \in (x/2, x)$ (you could conclude this by thinking about the Taylor series of sin, thinking about it's derivatives by themselves, or just by thinking about what the graph of sin looks like), so $\frac{\sin(x)}{x^{2/3}+\sin(x)} \leq \frac{x}{x^{2/3}+x/2}$. For $x \in [0,1]$, $x^{2/3} \geq x$, so $\frac{x}{x^{2/3}+x/2} \leq \frac{x}{3x/2} = \frac{2}{3}$. Hence, $\int_0^1 \frac{\sin(x)}{x^{2/3}+\sin(x)}\, dx \leq \frac{2}{3}$. Again, no divergence.

Avoiding divergence at ∞ is a little tricker. Since $\int_1^\infty \frac{\sin(x)}{x^{2/3}}$ converges, it suffices to prove $\int_1^\infty \frac{\sin(x)}{x^{2/3}+\sin(x)} - \frac{\sin(x)}{x^{2/3}}\, dx$ converges. The integrand can be rearranged to $\frac{-\sin^2(x)}{x^{2/3}(x^{2/3}+\sin(x))}$. Now, we are in a better situation, since the integrand is strictly non-positive. It suffices to prove that $\int_1^\infty \frac{\sin^2(x)}{x^{2/3}(x^{2/3}+\sin(x))}\, dx$ converges. But for $x \geq 1$ $\frac{1}{x^{2/3}(x^{2/3}+\sin(x))} \leq \frac{2}{x^{4/3}}$, so

$$\int_1^\infty \frac{\sin^2(x)}{x^{2/3}(x^{2/3}+\sin(x))}\, dx \leq 2\int_1^\infty \frac{\sin^2(x)}{x^{4/3}}\, dx \leq 2\int_1^\infty \frac{1}{x^{4/3}}\, dx.$$

The final integral converges, and we are done.