DAY 15 PROBLEMS AND SOLUTIONS

Exercise 1. For $s > \frac{1}{2}$, let $H^s(\mathbb{R}^n)$ denote the Sobolev space

$$H^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 \, d\mu(\xi) < \infty\}$$

(where μ is the Lebesgue measure and $\hat{f}$ is the Fourier transform of f). Show that if $u, v \in H^s(\mathbb{R}^n)$ for $s > n/2$, the $uv \in H^s(\mathbb{R}^n)$ and

$$||uv||_{H^s(\mathbb{R}^n)} \leq C||u||_{H^s(\mathbb{R}^n)}||v||_{H^s(\mathbb{R}^n)}$$

for a constant C depending only on s and n.

Solution 1. Note that $\hat{uv} = \hat{u} * \hat{v}$. Then

$$||uv||_{H^s}^2 = \int \left(\int (1+|\xi|^2)^{s/2} \hat{u}(\xi - \eta)\hat{v}(\eta) \, d\eta \right)^2 \, d\xi.$$

There exists a constant C sufficiently large so that $(1+|\xi|^2)^{s/2} \leq C((1+|\xi-\eta|^2)^{s/2} + (1+|\eta|^2)^{s/2})$. Plugging this inequality in, we can bound the previous integral above by

$$C\int \left(\int (1+|\xi-\eta|^2)^{s/2}\hat{u}(\xi-\eta)\hat{v}(\eta) + \int (1+|\eta|^2)^{s/2}\hat{u}(\xi-\eta)\hat{v}(\eta) \, d\eta \right)^2 = C||J^{s/2}\hat{u}*\hat{v}+\hat{u}*J^{s/2}\hat{v}||_{L^2}^2$$

where $J(a) = 1 + a^2$. We now see that $||uv||_{H^s}^2 \leq C||J^{s/2}\hat{u} * \hat{v} + \hat{u} * J^{s/2}\hat{v}||_{L^2}^2$, so $||uv||_{H^s} \leq ||J^{s/2}\hat{u} * \hat{v} + \hat{u} * J^{s/2}\hat{v}||_{L^2} \leq C||J^{s/2}\hat{u} * \hat{v}||_{L^2} + ||\hat{u} * J^{s/2}\hat{v}||_{L^2}$. By Young's inequality and the standard fact that $||\hat{v}||_{L^1} \leq ||v||_{H^s}$ if $s > n/2$, we see that $||J^{s/2}\hat{u}*\hat{v}||_{L^2} \leq ||J^{s/2}\hat{u}||_{L^2}||\hat{v}||_{L^1} \leq C||u||_{H^s}||v||_{H^s}$. Similarly, $||\hat{u} * J^{s/2}\hat{v}||_{L^2} \leq C||u||_{H^s}||v||_{H^s}$, so all together, we have $||uv||_{H^s} \leq C||u||_{H^s}||v||_{H^s}$, as desired.

Exercise 2. For $s > \frac{1}{2}$ let $H^s(\mathbb{R}^n)$ denote the Sobolev space

$$H^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 \, d\mu(\xi) < +\infty\}$$

(where μ is the Lebesgue measure and $\hat{f}$ is the Fourier transform of f). Use the Fourier transform to prove that if $u \in H^s(\mathbb{R}^n)$ for $s > n/2$, then $u \in L^\infty(\mathbb{R}^n)$, wih the bound

$$||u||_{L^\infty} \leq C||u||_{H^s(\mathbb{R}^n)}$$

for a constant C depending only on s and n.

Solution 2. Recall that $||u||_{L^\infty} \leq ||\hat{u}||_{L^1}$. Let's write $\hat{u}(\xi) = (1+|\xi|^2)^{s/2}(1+|\xi|^2)^{-s/2}\hat{u}(\xi)$. Then by Hölder's inequality, if we let $C = |(|1+|\xi|)^{-s/2}||_{L^2}$, which is finite because $s > n/2$, we see that

$$||\hat{u}||_{L^1} \leq ||(1+|\xi|^2)^{-s/2}||_{L^2}||(1+|\xi|^2)^{s/2}\hat{u}(\xi)||_{L^2} = C||u||_{H^s(\mathbb{R}^n)}.$$

Hence, $||u||_{L^\infty} \leq C||u||_{H^s(\mathbb{R}^n)}$, as desired.

Exercise 3. Let $H^s(\mathbb{R})$ be the Sobolev space on $\mathbb{R}$ with the norm

$$||u||_{H^s}^2 = \int_{\mathbb{R}} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi.$$

Prove that for non-negative real numbers $r < s < t$, for any $\varepsilon > 0$, there exists $C > 0$ such that

$$||u||_{H^s} \leq \varepsilon ||u||_{H^t} + C||u||_{H^r} \text{ whenever } u \in H^t(\mathbb{R}).$$

Solution 3. Let's split up the integral $\int (1+|\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi$ into a "large frequency" subdomain and a "small frequency" subdomain (this is what you would do to prove that $L^s \subset L^t \cap L^r$, which is how I viewed this problem). Let's call the large frequency domain $L = \{\xi : 1 + |\xi|^2 \geq \eta\}$ and the small frequency domain $S = \{\xi : 1 + |\xi|^2 < \eta\}$. On L, $(1+|\xi|^2)^s = (1+|\xi|^2)^{s-r}(1+|\xi|^2)^r \leq \eta^{s-r}(1+|\xi|^2)^r$, so $\int_L (1+|\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi \leq \eta^{s-r} \int_L (1+|\xi|^2)^r |\hat{u}(\xi)|^2 \, d\xi \leq \eta^{s-r} ||u||_{H^r}^2$. Taking η sufficiently large, since $s - r < 0$, we can get $\eta^{s-r} < \varepsilon^2$, so $\int_L (1+|\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi \leq \varepsilon^2 ||u||_{H^r}^2$. Similarly, we see that $\int_S (1+|\xi|^2)^t |\hat{u}(\xi)|^2 \, d\xi \leq \eta^{s-t} \int_S (1+|\xi|^2)^r |\hat{u}(\xi)|^2 \, d\xi \leq \eta^{s-t} ||u||_{H^t}^2$. Putting this together, we see that $||u||_{H^s}^2 \leq \varepsilon^2 ||u||_{H^r}^2 + C^2 ||u||_{H^t}^2$, where $C^2 = \eta^{s-t}$. Then $||u||_{H^s} \leq \sqrt{\varepsilon ||u||_{H^r}^2 + C||u||_{H^t}^2} \leq \varepsilon ||u||_{H^r} + C||u||_{H^t}$, as desired.

Exercise 4. *Extra 721 Problem:*

Prove that if K is a subset of $\mathbb{R}^n$ such that every continuous real-valued function on K is bounded, then K is compact.

Solution 4. Suppose K is not compact. Then since it is a subset of $\mathbb{R}^n$, it is either not closed or not bounded. Suppose it is not closed. Take a sequence $\{x_n\}_{n=1}^{\infty}$ converging to some $y \in K$. Let $f(x) = \frac{1}{||x-y||}$. This is continuous except at y, and in particular on K, but it is not bounded, since $f(x_n) \to \infty$. If K is not bounded, take $f(x) = |x|$ on K. This is continuous but now bounded because K is not bounded.

Exercise 5. *Extra 721 Problem:*

Let $f : [0,1] \to \mathbb{R}$ be continuous with $\min_{0 \leq x \leq 1} f(x) = 0$. Assume that for all $0 \leq a \leq b \leq 1$, we have $\int_a^b [f(x) - \min_{a \leq y \leq b} f(y)] \, dx \leq \frac{|b-a|}{2}$. Prove that for all $\lambda \geq 0$, we have

$$|\{x : f(x) > \lambda + 1\}| \leq \tfrac{1}{2} |\{x : f(x) > \lambda\}|.$$

Solution 5. Let λ be arbitrary. Since f is continuous, $\{x : f(x) > \lambda\}$ is a disjoint union of intervals $I_1, I_2, \ldots$. Since $\{x : f(x) > \lambda + 1\} \subsetneq \{x : f(x) > \lambda\}$, $\bigcup_{n \in \mathbb{N}} (I_n \cap \{x : f(x) > \lambda + 1\}) = \{x : f(x) > \lambda + 1\}$. Now suppose $|I_n \cap \{x : f(x) > \lambda + 1\}| > |I_n|/2$. Then since $f(x) \geq \lambda$ for $x \in I_n$, $\int_{I_n} [f(x) - \min_{y \in I_n} f(y)] \, dx \geq \int_{I_n} [f(x) - \lambda] \, dx \geq |\{x : f(x) > \lambda + 1\}| > |I_n|/2$, a contradiction. Hence, $|I_n \cap \{x : f(x) > \lambda + 1\}| \leq |I_n|/2$ for all n. Summing, we see that $|\{x : f(x) > \lambda + 1\} \leq \frac{1}{2} |\{x : f(x) > \lambda\}|$.

Exercise 6. *Extra 721 Problem:*

Consider the sequence of function $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \int_0^n \frac{\sin(sx)}{\sqrt{s}} \, ds.$$

(a) Show that f_n converges uniformly as $n \to \infty$ on any interval (α, β) for $0 < \alpha < \beta < \infty$.

(b) Show that f_n does not converge uniformly on $(0, 1]$ as $n \to \infty$.

(c) Does f_n converge uniformly on $[1, \infty)$ as $n \to \infty$?

Solution 6.

(a) To prove that f_n converges uniformly on (α, ∞), we will prove that
$$\left|\int_N^\infty \frac{\sin(sx)}{\sqrt{s}}\, ds\right| \lesssim_\alpha N^{-1/2}.$$
Integrating by parts in $\int_N^\infty \frac{\sin(sx)}{\sqrt{s}}\, ds$ gives (up to some absolute constants) $\frac{1}{x}\left(\frac{\cos(sx)}{\sqrt{s}}|_N^\infty + \int_N^\infty \frac{\cos(sx)}{s^{3/2}}\right)$. This has magnitude $\lesssim \frac{1}{x\sqrt{N}} \lesssim_\alpha \frac{1}{\sqrt{N}}$, as desired. If f_n converges on a set U and $V \subset U$, then f_n converges uniformly on V, so we have uniform convergence on (α, β) as well.

(b) We will prove that for any N, $\int_N^\infty \frac{\cos(s/N)}{\sqrt{s}}\, ds \gtrsim \sqrt{N}$. Substituting $u = s/N$ in that integral gives $N\int_1^\infty \frac{\cos(u)}{\sqrt{Nu}}\, du = \sqrt{N}\int_1^\infty \frac{\cos(u)}{\sqrt{u}}\, du \gtrsim \sqrt{N}$. It follows that $f_n(x)$ does not converge uniformly on $(0, 1]$.

(c) This is what we proved in (a).

Exercise 7. *Extra 721 Problem:* For $c_k \in \mathbb{R}$, say that $\prod_k c_k$ convergences if $\lim_{K\to\infty} \prod_{k=1}^K c_k = C$ exists for $C \neq 0, \infty$.

(a) Prove that if $0 < a_k < 1$ for all k, or if $-1 < a_k < 0$, for all k, then $\prod_k (1 + a_k)$ converges if and only if $\sum_k a_k$ conveges.

(b) However, prove that $\prod_k (1 + \frac{(-1)^k}{\sqrt{k}})$ diverges.

Solution 7. Taking the logarithm of both sides, we see that in the given range for a_k, $\lim_{K\to\infty} \prod_{k=1}^K (1 + a_k) = C \in (-\infty, \infty)$ if and only if $\lim_{K\to\infty} \sum_{k=1}^K \log(1 + a_k) = \log(C) \in (-\infty, \infty)$. Hence, it suffices to prove $\sum_{k=1}^\infty \log(1 + a_k)$ converges if and only if $\sum_{k=1}^\infty a_k$ converges.

If $a_k > 0$, then $\log(2)a_k < \log(1 + a_k) < a_k$, so $\sum_{k=1}^\infty \log(1 + a_k)$ converges if and only if $\sum_{k=1}^\infty a_k$ converges.

If $a_k < 0$, note that $a_k < -\frac{1}{2}$ can only happen finitely often if either sum converges. Removing finitely many terms will not change the convergence of either sum, so we may assume without loss of generality that $a_k \geq -\frac{1}{2}$ for all k. Then $-2\log(1/2)a_k < \log(1+a_k) < a_k$. Again, $\sum_{k=1}^\infty \log(1 + a_k)$ converges if and only if $\sum_{k=1}^\infty a_k$ converges.

For $\prod_{k=1}(1 + \frac{(-1)^k}{\sqrt{k}})$, note that the product converges only if the product
$$\prod_{j=1}\left(1 - \frac{1}{\sqrt{2j}}\right)\left(1 + \frac{1}{\sqrt{2j+1}}\right) = \prod_{j=1}\left(1 + \frac{\sqrt{2j+1} - \sqrt{2j} - 1}{\sqrt{2j}\sqrt{2j+1}}\right)$$
converges. Using the fact that $\sqrt{x+1} - \sqrt{x} \leq \frac{1}{2\sqrt{x}}$, we see that $\frac{\sqrt{2j+1}-\sqrt{2j}-1}{\sqrt{2j}\sqrt{2j+1}} \leq \frac{-1}{2(2j+1)}$. But $\sum_j \frac{-1}{2(2j+1)}$ certainly does not converge, so neither does $\prod_{j=1}\left(1 - \frac{1}{\sqrt{2j}}\right)\left(1 + \frac{1}{\sqrt{2j+1}}\right)$, nor $\prod_{k=1}(1 + \frac{(-1)^k}{\sqrt{k}})$.