DAY 14 PROBLEMS

Exercise 1. The following distributions u, v on $\mathbb{R}^2$ are defined by pairing with Schwartz functions via

$$\langle u, \phi\rangle = \int_0^2 \phi(0,t)\ dt$$

$$\langle v, \phi\rangle = \int_0^2 \phi(t,0)\ dt$$

Show that the convolution $u*v$ can be identified with a finite, absolutely continuous measure μ. Find $g \in L^1(\mathbb{R}^2)$ such that $\int \phi\ d\mu = \int \phi g\ dx$ for all Schwartz functions ϕ.

Solution 1. The primary difficult in this problem for me was remembering what the convolution of two distributions is. In general, if you want to figure out some sort of property for distributions, go to a different page and see if you can figure out the property assuming your distribution is a smooth function. Once you get the property for smooth functions, assume it works for all distributions, then go to the page with the problem and write the property down without proof like you knew it all along.

In this case, if we imagined u and v to be smooth functions, then applying the usual convolution formula and changing variables, we get $\langle u*v, \phi\rangle = \int u(x)v(y)\phi(x+y)\ dx\ dy$. This suggests that we should claim $\langle u*v, \phi\rangle = \langle u(x), \langle v(y), \phi(x+y)\rangle\rangle$, where $\langle v(y), \phi(x+y)\rangle$ is the map $x \mapsto \langle v(y), \phi(x+y)\rangle$. This is well-defined and characterizes the convolution of compactly supported distributions - it might not be how one would define the convolution, but I think you would be justified in immediately writing $\langle u*v, \phi\rangle = \langle u(x), \langle v(y), \phi(x+y)\rangle\rangle$.

With this definition in hand, the rest of the problem is fortunately pretty easy. We can compute $\langle v(y), \phi(x+y)\rangle = \int_0^2 \phi(t+x_1, x_2)\ dt$. This is a smooth function of x, call it $\psi(x)$. Then

$$\langle u(x), \psi(x)\rangle = \int_0^2 \psi(0,s)\ ds = \int_0^2\int_0^2 \phi(t,s)\ dt\ dx = \langle \chi_{[0,2]\times[0,2]}, \phi\rangle.$$

Hence, $\langle u*v, \phi\rangle = \langle g, \phi\rangle$, where $g(x) = \chi_{[0,2]\times[0,2]} \in L^1(\mathbb{R}^2)$, and we can identify $u*v$ with $g(x)\ dx$, which is a finite, absolutely continuous measure.

Exercise 2. Let $\mathcal{D}'(\mathbb{R})$ denote the space of distributions on $\mathbb{R}$ with the weak-$*$ topology. Determine the limit in $\mathcal{D}'(\mathbb{R})$ of the sequence of functions in $\mathbb{R}$:

$$\lim_{n\to\infty} \sqrt{n}e^{\frac{i}{2}nx^2}.$$

Solution 2. Let $u_n = \sqrt{n}e^{inx^2/2}$. For $f \in C_c^\infty(\mathbb{R})$, we have

$$\begin{aligned}\langle \hat{u}_n, f\rangle &= \langle u_n, \hat{f}\rangle \\ &= \int \sqrt{n}e^{inx^2/2}\hat{f}\ dx \\ &= \int\int \sqrt{n}e^{inx^2/2}e^{-iux}f(u)\ du\ dx \\ &= \int\int \sqrt{n}e^{in/2(x-u/n)^2}e^{-iu^2/(2n)}f(u)\ dx\ du \\ &= \int \sqrt{n}e^{iny^2/2}\ dy \int e^{-iu^2/(4n)}f(u)\ du \\ &= \int e^{iy^2}\ dy \int e^{-iu^2/(4n)}f(u)\ du.\end{aligned}$$

Since f is C_c^∞, we can use dominated convergence to conclude the final integral is $\int f(u)\,du = \hat{f}(0)$. We are treating u_n as tempered distribution here, which is justified since each u_n has at most polynomial growth. For now, let us denote $\int e^{iy^2}\ dy = C$. We conclude that $\hat{u}_n \to C\hat{\delta}_0$, so $u_n \to C\delta_0$. To find the exact value of C, you need to be carefuly about how you normalize Fourier transform and then compute $\int e^{iy^2}\ dy = (1+i)\sqrt{\pi/2}$. That computation is likely beyond the scope of the exam (the usual way to do it is contour integration), so I think you are fine to leave it as a constant.

Exercise 3. A real-valued function f defined on $\mathbb{R}$ belongs to the space $C^{1/2}(\mathbb{R})$ if and only if

$$\sup_{x\in\mathbb{R}}|f(x)| + \sup_{x\neq y}\frac{|f(x)-f(y)|}{\sqrt{|x-y|}} < \infty.$$

Prove that a function $f \in C^{1/2}(\mathbb{R})$ if and only if there is a constant C so that for every $\varepsilon > 0$, there is a bounded function $\phi \in C^\infty(\mathbb{R})$ such that

$$\sup_{x\in\mathbb{R}}|f(x)-\phi(x)| \le C\sqrt{\varepsilon} \text{ and } \sup_{x\in\mathbb{R}}\sqrt{\varepsilon}|\phi'(x)| \le C.$$

Solution 3. Suppose $f \in C^{1/2}(\mathbb{R})$. Let φ be a smooth bump function, symmetric about the $x = 0$, supported on $[-1,1]$, with $||\varphi(x)||_{L^1} = 1$. Let $\varphi_\varepsilon(x) = \frac{\varphi(x/\varepsilon)}{\varepsilon}$. Note that φ_ε is supported on $[-\varepsilon,\varepsilon]$ and $||\varphi_\varepsilon||_{L^1} = 1$. Define $\phi = f * \varphi_\varepsilon$. Let's first check that $\sup_{x\in\mathbb{R}}|f(x)-\phi(x)| \le C\sqrt{\varepsilon}$. We see that

$$|f(x)-\phi(x)| \le \int_{\mathbb{R}}\varphi_\varepsilon(y)|f(x)-f(x-y)|\ dy \le \int_{-\varepsilon}^{\varepsilon}\varphi_\varepsilon(y)\sqrt{y}\ dy.$$

By Hölder's inequality, $\int_{-\varepsilon}^{\varepsilon}\varphi_\varepsilon(y)\sqrt{y}\ dy \le ||\varphi_\varepsilon||_{L^1}||\sqrt{y}||_{L^\infty([-\varepsilon,\varepsilon])} \le \sqrt{\varepsilon}$.

For the second part, we have $\phi'(x) = \frac{1}{\varepsilon^2}\int_{-\infty}^{\infty} f(x-y)\varphi'(y/\varepsilon)\ dy$. We know that by the assumption that φ is symmetric that about $x = 0$ that φ' is odd, so we can write this as $\frac{1}{\varepsilon^2}\int_0^\infty [f(x-y)-f(x+y)]\varphi'(y/\varepsilon)\ dy$. We know $\varphi'(y)$ is supported on $[-\varepsilon,\varepsilon]$, so $|f(x-y)-f(x+y)| \le \sqrt{2y} \le C_1\sqrt{\varepsilon}$ for some $C_1 > 0$. We also have that $\varphi'(y)$ is uniformly bounded by some $C_2 > 0$. Then

$$\frac{1}{\varepsilon^2}\int_0^\infty [f(x-y)-f(x+y)]\varphi'(y/\varepsilon)\ dy \le \frac{1}{\varepsilon^2}\int_0^\varepsilon C_1\sqrt{\varepsilon}C_2\ dy = \frac{C_1C_2}{\sqrt{e}}.$$

Hence, for $C = C_1C_2$ $|\phi'(x)| \le \frac{C}{\sqrt{\varepsilon}}$, as desired.

Now suppose that there is a constant C so that for any $\varepsilon > 0$, we can find $\phi \in C^\infty(\mathbb{R})$ such that $\sup_{x\in\mathbb{R}} |f(x) - \phi(x)| \le C\sqrt{\varepsilon}$ and $\sup_{x\in\mathbb{R}} \sqrt{\varepsilon}|\phi'(x)| \le C$. First, let's prove that $\sup_{x\in\mathbb{R}} |f(x)|$ is bounded. Take $\phi \in C^\infty(\mathbb{R})$ with maximum M such that $\sup_{x\in\mathbb{R}} |f(x) - \phi(x)| < C$. Then by the triangle inequality, $\sup_{x\in\mathbb{R}} |f(x)| < C + M$. Now, let's prove $\sup_{x\neq y} \frac{|f(x)-f(y)|}{\sqrt{|x-y|}} < \infty$. Take $x \neq y \in \mathbb{R}$, set $\varepsilon = x - y$, and choose ϕ corresponding to that value of ε. Then

$$\frac{|f(x) - f(y)|}{|x - y|} < \frac{|f(x) - \phi(x)|}{\sqrt{|x - y|}} + \frac{|\phi(x) - \phi(y)|}{\sqrt{|x - y|}} + \frac{|\phi(y) - f(y)|}{\sqrt{|x - y|}} \le C\frac{\varepsilon}{|x - y|} + \frac{C|x - y|}{\sqrt{\varepsilon|x - y|}} + C\frac{\varepsilon}{|x - y|} = 3C.$$

Hence, $\sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|} < \infty$, so $f \in C^{1/2}(\mathbb{R})$.

Exercise 4. Let $H^1([0, 1]) = \{f \in L^2([0, 1]) : f' \in L^2\}$, where f' denotes the distributional derivative of f. Equip H^1 with the norm $||f||_{H^1} = ||f||_{L^2} + ||f'||_{L^2}$.

For $\alpha \in [0, 1]$, denote $||f||_{C^\alpha} = \sup_{x\in[0,1]} |f(x)| + \sup_{x\neq y\in[0,1]} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$ and $C^\alpha([0, 1]) = \{f \in C([0, 1]) : ||f||_{C^\alpha} < \infty\}$.

You may use without proof that H^1 and C^α are both Banach spaces.

(1) Prove that $H^1([0, 1]) \subset C^{1/2}([0, 1])$.
(2) Prove that the closed unit ball in $H^1([0, 1])$ is compact in $C^\alpha([0, 1])$ for any $\alpha < 1/2$.
(3) Is the closed unit ball in $H^1([0, 1])$ compact in $C^{1/2}([0, 1])$? Prove or give a counterexample.

Solution 4.

(1) Take $f \in H^1([0, 1])$. Formally, we want to say that $|f(y) - f(x)| = |\langle f, \delta_y - \delta_x\rangle| = |\langle f', \chi_{[x,y]}\rangle| \le ||f'||_{L^2}|x - y|^{1/2} \le ||f||_{H^1}|x - y|^{1/2}$. For $f \in C^\infty([0, 1]) \subset H^1([0, 1])$, we see that this is true just using standard facts about distributions (in particular, that $\chi'_{[x,y]} = \delta_y - \delta_x$, which follows from the fundamental theorem of calculus). Smooth functions are dense in $H^1([0, 1])$ (I would suggest on the qual assuming that smooth functions are dense in every function space, then going back and proving that is the case if you have time. In this case, it follows fairly easily by an approximation of the identity argument, you can google it if you are curious), so $f \in H_1$, let $\varphi_n \in C^\infty([0, 1])$ be a sequence with $\varphi_n \to f$ in the H^1 norm. It follows that $|(\varphi_n - f)(y) - (\varphi_n - f)(x)| \le ||\varphi_n - f||_{H^1}|x - y|^{1/2}$. By the triangle inequality, we see that

$$\begin{aligned} |f(y) - f(x)| &\le |(\varphi_n - f)(y) - (\varphi_n - f)(x)| + |\varphi_n(y) - \varphi_n(x)| \\ &\le (||\varphi_n - f||_{H^1} + ||\varphi_n||_{H^1})|x - y|^{1/2} \\ &\to ||f||_{H^1}|x - y|^{1/2} \end{aligned}$$

Hence $|f(y) - f(x)| \le ||f||_{H^1}|x - y|^{1/2}$, so $\sup_{x\neq y\in[0,1]} \frac{|f(x)-f(y)|}{|x-y|^{1/2}} \le ||f||_{H^1}$. We also see that f is necessarily continuous, a nice thing not to have to also check and justifying considering the pointwise behavior of f.

Now suppose $\sup_{x\in[0,1]} |f(x)| > 2||f||_{H^1}$. Then for some x_0, $f(x_0) > 2||f||_{H^1}$. We know that $|f(x_0) - f(y)| \le ||f||_{H^1}|x - y|^{1/2}$, so $|f(y)| > 2||f||_{H^1} - ||f||_{H^1}|x_0 - y|^{1/2} \ge ||f||_{H^1}$. Therefore, $\int_0^1 |f(x)|^2\, dx > ||f||^2_{H^1} \ge ||f||^2_{L^2} = \int_0^1 |f(x)|^2\, dx$, a contradiction.

Hence, $\sup_{x\in[0,1]}|f(x)| < 2||f||_{H^1}$, so putting this together, we have $f \in C^{1/2}([0,1])$, with $||f||_{C^{1/2}([0,1])} \le C||f||_{H^1}$ for a constant $C > 0$.

(2) By the previous problem, we know that the unit ball in H^1 is contained in the unit ball in $C^{1/2}$. We will prove that the $C^{1/2}$ unit ball B is compact in $C^\alpha([0,1])$ for any $\alpha < 1/2$. Take $f_n \in B$. By Arzela-Ascoli (using the fact that $f_n \in B$ to obtain both uniform boundedness and equicontinuity), f_n has a uniformly convergent subsequence $f_{n_k} \to g$. Since f_{n_k} converges uniformly, it is Cauchy in the sup-norm. We also know that

$$\begin{aligned}\sup_{x,y\in(0,1)} \frac{|(f_{n_k}-f_{n_j})(x)-(f_{n_k}-f_{n_j})(y)|}{|x-y|^\alpha} &= \sup_{x,y\in(0,1)} \left(\frac{|(f_{n_k}-f_{n_j})(x)-(f_{n_k}-f_{n_j})(y)|^{1/(2\alpha)}}{|x-y|^{1/2}}\right)^{2\alpha} \\ &= \sup_{x,y\in(0,1)} \left(\frac{|(f_{n_k}-f_{n_j})(x)-(f_{n_k}-f_{n_j})(y)|}{|x-y|^{1/2}}\right)^{2\alpha} \\ &\times \left(\sup_{x,y\in[0,1]} (|f_{n_k}-f_{n_j}|(x)+|f_{n_k}-f_{n_j}|(y))\right)^{1-2\alpha} \\ &\le C||f_{n_k}-f_{n_j}||_{\sup}^{1-2\alpha}\end{aligned}$$

Therefore, f_{n_k} is Cauchy in the C^α norm, so since C^α is complete, f_{n_k} converges in C^α, as desired.

(3) It is not. For a counterexample, suppose f_n is a triangle with base of width $2n^{-2}$ centered at the origin and height n^{-1}. Then f_n' (distributionally) is $n(\chi_{[-n^{-2},0]} - \chi_{[0,n^{-2}]})$, which is bounded in L^2 (f itself goes to 0 in L^2). So f_n is a bounded sequence in H^1. On the other hand, if $m > n$, $||f_n - f_m||_{C^\alpha} \gtrsim 1$, so f_n cannot have a convergent sequence in C^α. *AN: I know this is vague - I'll try to fill it in with more detail later*

Exercise 5. *Extra 721 Problem:*

Show that there is no sequence $\{a_n\}_{n\in\mathbb{N}}$ of positive numbers such that $\sum_{n\in\mathbb{N}} a_n|c_n| < \infty$ if and only if c_n is bounded.

Hint: Suppose such a sequence exists and consider the map $T : \ell^\infty(\mathbb{N}) \to \ell^1(\mathbb{N})$ given by $[Tf]_n = a_n f(n)$. The set of f such that $f(n) = 0$ for all but finitely many n is dense in ℓ^1 but not in ℓ^∞.

Solution 5. Suppose such a sequence exists and define T as in the problem. Then by the uniform boundedness principle, $T : \ell^\infty(\mathbb{N}) \to \ell^1(\mathbb{N})$ is continuous. It also maps the set $A = \{\{a_n\}_{n=0}^\infty : a_n = 0 \text{ for all but finitely many } n\}$ bijectively to itself. Finally, it is surjective, since if $\{b_n\} \in \ell^1(\mathbb{N})$ but $\{b_n/a_n\}$ is unbounded, then $\sum_n b_n/a_n a_n$ cannot converge, and hence we must have that $\{b_n/a_n\}$ is bounded. Then it is an open mapping, by the open mapping theorem. Take an open set $O \subset \ell^\infty(\mathbb{N})$ not intersecting A (for example, $O = \{\{x_n\} : x_n > 1 \text{ for all } n\}$). Then $T(O)$ is open in $\ell^1(\mathbb{N})$, so it must intersect K, and hence there exists $x \in \ell^\infty(\mathbb{N}) \setminus K$ such that $T(x) \in K$, which is a contradiction, since $T(x)$ has the same number of zero entries as x.

Exercise 6. *Extra 721 Problem:*

Let $C([0,1])$ denote the set of continuous functions on $[0,1]$ equipped with the sup-norm. Prove that there exists a dense subset of $C([0,1])$ consisting of functions are nowhere differentiable.

Solution 6. Define $A_n = \{f :$ there exists $x \in [0,1]$ such that $|f(x) - f(y)| \leq n|x - y|$ for all $y \in [0,1]\}$. Let's prove that A_n is closed and nowhere dense. Let f_m be a sequence in A_n converging to some f. Let x_m be a point such that $|f_m(x_m) - f_m(y)| \leq n|x_m - y|$ for all $y \in [0,1]$. Since x_m is a bounded sequence, it has a convergent subsequence $x_{m_j} \to x$. Then $|f(x) - f(y)| = \lim_{j\to\infty} |f_{m_j}(x_{m_j}) - f_{m_j}(y)| \leq \lim_{j\to\infty} n|x_{m_j} - y| = n|x - y|$, so $f \in A_n$ and hence A_n is closed. Suppose $f \in A_n$. Fix $\varepsilon > 0$. There exists δ such that for all $|h| < \delta$, $|f(x+h) - f(x)| < \varepsilon$. Let $g(y)$ be a continuous function from $[0,1]$ to $[-\varepsilon, \varepsilon]$ which is $-\varepsilon$ at x and ε at $x + c$, where c is sufficiently large that $c < \min(\delta, 1 - x, \varepsilon/(2n))$. Set $f_\varepsilon(y) = f(y) + g(y)$. Then $|f_\varepsilon - f| = |g| < \varepsilon$ and

$$|f_\varepsilon(x) - f_\varepsilon(x+c)| = |f(x) - f(x+c) + 2\varepsilon| \geq |2\varepsilon - \varepsilon| = \varepsilon.$$

Then $|f_\varepsilon(x) - f_\varepsilon(x+c)| \geq \varepsilon > nc$, so $f_\varepsilon \notin A_n$. It follows that A_n has empty interior. The intersection of A_n^c is an intersection of open dense sets, so it is residual and, in particular, non-empty. Suppose f in that intersection and f is differentiable at some point x with derivative α. Set $k = \lceil 2\alpha \rceil$. Then there exists some $\delta > 0$ such that for $y \in (x - \delta, x + \delta)$, $|f(y) - f(x)| \leq k|x - y|$. The function $y \mapsto \frac{|f(y)-f(x)|}{\delta}$ is continuous on the compact set $[0,1] \setminus (x - \delta, x + \delta)$ and hence achieves some maximum $\leq m$. Set $n = \max(k, m)$. We know $|f(y) - f(x)| \leq k|x - y| \leq n|x - y|$. Since $\frac{|f(y)-f(x)|}{\delta} \leq m \leq n$ for $y \in [0,1] \setminus (x - \delta, x + \delta)$, $|f(y) - f(x)| \leq n\delta \leq n|x - y|$. Then $f \in A_n$, a contradiction. Thus, f is nowhere differentiable, as required.

Exercise 7. *Extra 721 Problem:*

Let H be a Hilbert space. For a linear space $Y \subset H$, define $Y^\perp = \{x \in H : (x, y) = 0\}$.

(1) Prove that if Y is closed, then $Y^\perp$ is a closed linear subspace of H.
(2) Prove that for any $x \in H$, a minimizing sequence for $\inf_{y\in Y} |x - y|$ is Cauchy. Conclude that we can uniquely write $x = x^{||} + x^\perp$ with $x^{||} \in Y$ and $x^\perp \in Y^\perp$.
(3) Prove that if $f : H \to \mathbb{R}$ is bounded and linear, then there exists $y \in H$ such that $f(x) = (x, y)$ for all x.

Solution 7.

(1) Take a convergent sequence $y_n \to y$ with $y_n \in Y^\perp$ for all n. Then $(y, x) = \lim_{n\to\infty}(y_n, x) = 0$ for any $x \in Y$. Hence, $Y^\perp$ is closed. For any $\alpha \in \mathbb{R}$ and $x, y \in Y^\perp$ and all $z \in Y$, $(\alpha x + y, z) = \alpha(x, z) + (y, z) = 0$. Therefore, $Y^\perp$ is a vector space.
(2) Let x_n be a minimizing sequence for $d = \inf_{x\in Y} ||x - y||_H$, that is $\lim_{n\to\infty} ||x_n - y||_H = \inf_{x\in Y} |x - y|$ and $x_n \in Y$ for all n (I switched the letter's around when I wrote the solution, sorry). Let's prove this is Cauchy. We have that $||x_n - x_m||_H^2 = ||x_n||^2 + ||x_m||^2 - 2\langle x_n, x_m\rangle$. We somehow have to use that $||x_n - y||_H^2$ is a minimizing sequence to make $2\langle x_n, x_m\rangle \to ||x_n||_H^2 + ||x_m||_H^2$. Since $||x_n - x_m||_H^2 \geq 0$, we know $||x_n||_H^2 + ||x_m||_H^2 \geq 2\langle x_n, x_m\rangle$, we will prove that in the limit, the opposite inequality holds as well. We can make this pop out by computing the distance between $x_n + x_m$ and y:

$$||\frac{x_n + x_m}{2} - y||_H^2 = \frac{||x_n||_H^2}{4} + \frac{||x_m||_H^2}{4} + \frac{\langle x_n, x_m\rangle}{2} + ||y||_H^2 - \langle x_n, y\rangle - \langle x_m, y\rangle.$$

On the other hand, for any $\varepsilon > 0$, we can choose N sufficiently large such that for any $n, m \geq N$, $||x_n - y||^2 \leq d^2 + \varepsilon \leq ||\frac{x_n + x_m}{2} - y||_H^2 + \varepsilon$. Then $\frac{||x_n - y||^2}{2} + \frac{||x_m - y||^2}{2} \leq ||\frac{x_n + x_m}{2} - y||_H^2 + \varepsilon$. Expanding both sides and simplifying, we see that $4\varepsilon + 2\langle x_n, x_m\rangle \geq ||x_n||^2 + ||x_m||^2$. The sending $\varepsilon \to 0$ and using the fact from before that $||x_n||_H^2 + ||x_m||_H^2 \geq 2\langle x_n, x_m\rangle$ for all n, m, we see that $||x_n - x_m||_H^2 \to 0$, and hence x_n is Cauchy. Denote it's limit by $x^{||}$.

Since each $x_n \in Y$ and Y is closed, $x^{||} \in Y$. Let $x^{\perp} = y - x^{||}$. Take $z \in Y$. We have that at $t = 0$, $\frac{d}{dt}||y - x^{||} + tz||_H^2 = 2\langle y, z\rangle - 2\langle x^{||}, z\rangle = 0$, since the minimum of $||y - x^{||} + tz||_H^2$ is at 0. It follows that $\langle y - x^{||}, z\rangle = 0$, so $\langle x^{\perp}, z\rangle = 0$. Hence, $x^{\perp} \in Y^{\perp}$.

To see that the decomposition is unique, suppose we can find a pair $x_0 \in Y$, $x_1 \in Y^{\perp}$ such that $x_0 + x_1 = y$. Then $x_0 - x^{||} = x_1 - x^{\perp} \in Y \cap Y^{\perp}$. But if $z \in Y \cap Y^{\perp}$, then $(z, z) = 0$, so $z = 0$ and hence, $x_0 = x^{||}, x_1 = x^{\perp}$. Thus, we have uniqueness.

(3) If $f \equiv 0$, then we can take $y = 0$. Otherwise, let $Y = \ker(f)$. Since $f \neq 0$, $Y \neq H$, so there exists $\tilde{y} \neq 0 \in Y^{\perp}$. It follows from the last problem that $Y \cap Y^{\perp} = \{0\}$, so for $x_1, x_2 \in Y^{\perp}$, $f(x_1), f(x_2) \neq 0$. Then we can find $\alpha \neq 0$ such that $\alpha f(x_1) + f(x_2) = f(\alpha x_1 + x_2) = 0$, and hence $\alpha x_1 + x_2 \in Y \cap Y^{\perp}$, so $\alpha x_1 + x_2 = 0$. Since $Y^{\perp}$ therefore cannot contain a pair of linearly independent vectors, we have $\dim(Y^{\perp}) = 1$. So take some element $y \in Y^{\perp}$. Then there exists $\alpha \neq 0$ such that $f(y) = \alpha$, so $f(y) = \frac{\alpha}{||y||^2}(y, y)$. Set $\tilde{y} = \frac{\alpha y}{||y||^2}$, so that $f(y) = (y, \tilde{y})$. For all $x \in Y^{\perp}$, $x = \beta y$ for some $\beta \neq 0$, so

$$f(x) = f(\beta y) = \beta f(y) = \beta(y, \tilde{y}) = (\beta y, \tilde{y}) = (x, \tilde{y}).$$

For general $x \in H$, we use the previous problem to write $x = x^{||} + x^{\perp}$, with $x^{\perp} \in Y^{\perp}$ and $x^{||} \in Y$. Then $f(x^{||} + x^{\perp}) = f(x^{\perp}) = (x^{\perp}, \tilde{y}) = (x^{\perp} + x^{||}, \tilde{y})$, using the fact that $(y, x^{||}) = 0$ and $x^{||} \in \ker(f)$.