# DAY 13 PROBLEMS AND SOLUTIONS

**Exercise 1.** Suppose $g_n \in S(\mathbb{R}^2)$ and $\lim_{n\to\infty} ||g_n||_{L^2}(\mathbb{R}^2) = 0$. Show that there are $f_n \in C^2(\mathbb{R}^2)$ such that $\Delta f_n = f_n + g_n$ and $f_n$ satisfies

(1) $\lim_{n\to\infty} f_n(0,0) = 0$.
(2) $\lim_{n\to\infty} ||\partial^2_{x_1 x_2}(f_n)||_{L^2(\mathbb{R}^2)} = 0$.

**Solution 1.** Before we prove the properties, let's solve the equation $\Delta f_n = f_n + g_n$. Take the Fourier transform of both sides. Hopefully you recall that $\widehat{\Delta f_n}(\xi) = -|\xi|^2 \hat{f}_n(\xi)$, so the desired equation becomes $|\xi|^2 \hat{f}_n(\xi) + \hat{f}_n(\xi) = -\hat{g}(\xi)$, or in other words, $\hat{f}_n = -\frac{\hat{g_n}(\xi)}{1+|\xi|^2}$. Since $g_n$ is Schwartz, $\hat{f}_n$ is Schwartz as well, and hence it's Fourier inverse $f_n$ is a Schwartz function and hence in $C^2$. By construction, these functions $f_n$ satisfy $\Delta f_n = f_n + g_n$.

(1) Let $C = ||\frac{1}{1+|\xi|^2}||_{L^2}(\mathbb{R}^2)$. Then

$$f_n(0) = -\int e^{i0\cdot\xi} \frac{\hat{g_n}(\xi)}{1+|\xi|^2}\, dx = -\int \frac{\hat{g_n}(\xi)}{1+|\xi|^2}\, dx \leq ||\hat{g_n}||_{L^2} ||\frac{1}{1+|\xi|^2}||_{L^2} = C||\hat{g_n}||_{L^2} = C||g_n||_{L^2}.$$

Since $||g_n||_{L^2} \to 0$, $f_n(0) \to 0$ as well.

(2) By Plancheral's theorem,

$$||\partial^2_{x_1 x_2} f_n||_{L^2(\mathbb{R}^2)} = ||\xi_1\xi_2 \hat{f}_n||_{L^2(\mathbb{R}^2)} = ||\frac{\xi_1\xi_2}{1+|\xi|^2}\hat{g}_n||_{L^2(\mathbb{R}^2)}$$

By Hölder's inequality,

$$||\frac{\xi_1\xi_2}{1+|\xi|^2}\hat{g}_n||_{L^2(\mathbb{R}^2)} \leq ||\frac{\xi_1\xi_2}{1+|\xi|^2}||_{L^\infty(\mathbb{R}^2)} ||\hat{g}_n||_{L^2(\mathbb{R}^2)} = ||\frac{\xi_1\xi_2}{1+|\xi|^2}||_{L^\infty(\mathbb{R}^2)} ||g_n||_{L^2(\mathbb{R}^2)}.$$

Since $(\xi_1 - \xi_2)^2 \geq 0$, $\xi_1^2 + \xi_2^2 \geq 2\xi_1\xi_2$, and hence $\frac{\xi_1\xi_2}{1+\xi_1^2+\xi_2^2} < 1$. Thus, $||\frac{\xi_1\xi_2}{1+|\xi|^2}||_{L^\infty(\mathbb{R}^2)} < 1$, so $||\partial^2_{x_1x_2} f_n||_{L^2(\mathbb{R}^2)} \leq ||g_n||_{L^2}$. Since $||g_n||_{L^2} \to 0$, $||\partial^2_{x_1x_2} f_n||_{L^2(\mathbb{R}^2)} \to 0$ as well.

**Exercise 2.** Show that for $\alpha \neq 0$,

$$\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{\alpha}{\alpha^2 + n^2} = \frac{e^{2\pi\alpha} + 1}{e^{2\pi\alpha} - 1}.$$

*Hint: Apply the Poisson summation formula to the function $f(x) = e^{-c|x|}$, for an appropriate choice of $c$.*

**Solution 2.** First, let's compute the Fourier transform of $f_c(x) = e^{-2\pi c|x|}$ (using $2\pi c$ instead of $c$ will be convenient for what comes next). We see that

$$\begin{aligned}\hat{f}(\xi) &= \int e^{-2\pi i x\xi} e^{-2\pi c|x|}\, dx \\ &= \int_0^\infty e^{-2\pi x(i\xi + c)}\, dx + \int_{-\infty}^0 e^{-2\pi x(2\pi i\xi - c)}\, dx \\ &= \frac{1}{2\pi}\left(\frac{1}{i\xi + c} - \frac{1}{i\xi - c}\right) \\ &= \frac{1}{\pi}\frac{c}{\xi^2 + c^2}\end{aligned}$$

This looks pretty promising. The Poisson summation formula tells us that

$$\sum_{n\in\mathbb{Z}} e^{-2\pi c|n|} = \frac{1}{\pi}\sum_{n\in\mathbb{Z}} \frac{c}{n^2 + c^2}.$$

Let's set $c = \alpha$ and prove that $\sum_{n\in\mathbb{Z}} e^{-2\pi\alpha|n|} = \frac{e^{2\pi\alpha}+1}{e^{2\pi\alpha}-1}$. To do so, we will break up the sum and then apply the geometric series formula:

$$\sum_{n\in\mathbb{Z}} e^{-2\pi\alpha|n|} = 2\sum_{n=0}^{\infty} e^{-2\pi\alpha n} - 1 = \frac{2}{1 - e^{-2\pi\alpha}} - 1 = \frac{1 + e^{-2\pi\alpha}}{1 - e^{-2\pi\alpha}} = \frac{e^{2\pi\alpha} + 1}{e^{2\pi\alpha} - 1}.$$

**Exercise 3.** Find the Fourier transform of the following function: $f \in \mathbb{R}^2$:

$$f(x) = e^{ix\xi_0}|x - x_0|^{-1}.$$

**Solution 3.** First, let's suppose $\xi_0 = x_0 = 0$. Then since $f$ is a radial function homogenous of degree $-1$, $\hat{f}$ is radial and homogenous of degree $(-2 - (-1)) = -1$. Since every radial, homogenous function of degree $m$ function is $C|x|^m$ for some constant $C$, it follows that $\hat{f}(\xi) = \frac{C}{|\xi|}$. We can find the constant by computing

$$\langle \frac{1}{|x|}, \widehat{e^{-|\xi|^2}} \rangle = \langle \widehat{\frac{1}{|x|}}, e^{-|\xi|^2} \rangle = \langle \frac{C}{|\xi|}, e^{-|\xi|^2} \rangle.$$

We can compute that the Fourier transform of $e^{-|\xi|^2}$ is $\sqrt{\pi}e^{-\pi^2x^2}$, so

$$\sqrt{\pi}\int_{\mathbb{R}^2} \frac{e^{-\pi^2x^2}}{|x|}\, dx = C\int_{\mathbb{R}^2} \frac{e^{-|\xi|^2}}{|\xi|}\, d\xi.$$

Changing variables, we see that $\sqrt{\pi}\int_{\mathbb{R}^2} \frac{e^{-\pi^2x^2}}{|x|}\, dx = \sqrt{\pi}\int_{\mathbb{R}^2} \frac{e^{-x^2}}{|x|}\, dx$ (the scale invariance seen here is a property of the scale invariant differential term $\frac{dx}{x} = d\log(x)$). Hence, $C = \sqrt{\pi}$.

**Exercise 4.** Find the spectrum of the linear operator $A$ in $L^2(\mathbb{R})$ defined as

$$(Af)(x) = \int_{-\infty}^{\infty} \frac{f(y)}{1 + (x - y)^2}\, dy.$$

(The spectrum of a linear operator $T$ is the closure of the set of all complex numbers $\lambda$ such that the operator $T - \lambda I$ does not have a bounded inverse. Hint: it may be helpful to find Fourier transform of $1/(1 + x^2)$.)

**Solution 4.** Let's start with the hint. We will prove that the Fourier transform of $\frac{1}{1+x^2}$ is $\pi e^{-|x|}$. It is not too hard to check that this is correct by Fourier inverting $\pi e^{-|x|}$ and splitting up the domain of the resulting integral, but actually coming up with that on your own would be tricky. Here is one approach. Let $g(\xi)$ denote the Fourier transform of $\frac{1}{1+x^2}$. Differentiating under the integral, we see that $\frac{\partial^2}{\partial\xi^2}g(\xi) = -\hat{1} + g(\xi)$. Of course, $\hat{1}$ only makes sense as a distribution, but with a bit of distribution theory and the fact that $\int f(x)\,dx = \check{f}(0)$, we see that it is $\delta_0$: for any $h \in C_c^\infty(\mathbb{R})$, $(\hat{1}, h) = (1, \hat{h}) = \check{\hat{f}}(0) = f(0)$.

Now we have a differential equation, which does something weird at 0, but on $(-\infty, 0)$ and $(0, \infty)$, we should be able to solve this as one normally would (which for me means plug in $e^{a\xi}$ and hope a solution falls out). In thise case, we are in luck: we see that any solution must be of the form $Ce^{a\xi}$ for $a = \pm 1$. Since $\frac{1}{1+x^2} \in L^1(\mathbb{R})$ and the Fourier transform of an $L^1$ function is in $C_0(\mathbb{R})$, we know that we must have $a = 1$ on $(-\infty, 0)$ and $a = -1$ on $(0, \infty)$ for $g$ to have good decay at $\infty$. For $g$ to be continuous at 0, we must have the same constant factor throughout, which must equal $g(0)$. It is straightforward to compute that $g(0) = \pi$, and we are left with $g(\xi) = \begin{cases} \pi e^{-\xi} & \xi \geq 0 \\ \pi e^{\xi} & \xi < 0 \end{cases} = \pi e^{-|\xi|}$.

Now that we have that out of the way, we return to the actual problem. Note that $Af = f * h$, where $h(x) = \frac{1}{1+x^2}$. Then $\hat{Af} = \hat{f}\hat{h} = \pi e^{-|x|}\hat{f}$, so $(Af - \lambda f)\hat{} = (\pi e^{-|x|} - \lambda)\hat{f}$. Then if $A - \lambda I$ has an inverse $G_\lambda : L^2 \to L^2$, then if $\hat{Gf}$ is well defined, it must equal $\frac{1}{\pi e^{-|x|} - \lambda}\hat{f}$. If $\lambda \notin [0, \pi]$, then by Hölder's inequality and Plancheral, $||\hat{Gf}||_{L^2} \leq \frac{1}{|\lambda| + \pi}||f||_{L^2}$. Then $\hat{Gf}$ is bounded $L^2 \to L^2$, so by Plancheral again, $Gf$ is as well. Note that $\hat{Af} = \pi e^{-|x|}\hat{f}$ has non-trivial kernel, since any $f$ with $\hat{f} \perp \pi e^{-|x|}$ will be sent to 0, so $Af$ has the same non-trivial kernel, and hence $0 \in \mathrm{spec}(A)$.

Finally, we will check that $(0, \pi] \in \mathrm{spec}(A)$. Take $\lambda \in (0, \pi]$ and choose $x_0$ such that $\pi e^{-|x_0|} = \lambda$ (there usually will be two such values, choose one). Take $f \in L^2$ such that $\hat{f} = \chi_{[x_0-1, x_0+1]}$. If $\hat{Gf}$ is bounded $L^2 \to L^2$, then $\frac{\chi_{[x_0-1,x_0+1]}(x)}{\pi e^{-|x|} - \lambda} \in L^2$, or in other words, $\int_{x_0-1}^{x_0+1} \frac{1}{|\pi e^{-|x|} - \lambda|^2}\,dx < \infty$, which would imply that $|\pi e^{-|x|} - \lambda| \gtrsim |x - x_0|^{1/2}$ as $x \to x_0$. But by the mean value theorem, we know that $|\pi e^{-|x|} - \lambda| = |\pi e^{-|x|} - \pi e^{-|x_0|}| \lesssim |x - x_0| \ll |x - x_0|^{1/2}$ as $x \to x_0$. Therefore, $\hat{Gf}$ cannot be bounded $L^2 \to L^2$ and hence neither can $Gf$. It follows that $\lambda \in \mathrm{spec}A$, and since $\lambda \in (0, \pi]$ was arbitrary, we see that $(0, \pi] \subset \mathrm{spec}(A)$. Putting this all together, we conclude that $\mathrm{spec}(A) = [0, \pi]$.

**Exercise 5.** *Extra 721 Problem:* For $n \geq 2$ an integer, define $F(n)$ to be the function $F(n) = \max\{k \in \mathbb{Z} : 2^k/k \leq n\}$. Does $\sum_{n=2}^\infty 2^{-F(n)}$ converge?

**Solution 5.** We will write

$$\sum_{n=2}^\infty 2^{-F(n)} = \sum_{k=2}^\infty \left( \sum_{n:F(n)=k} 2^{-k} \right) = \sum_{k=2}^\infty \#\{n : F(n) = k\}2^{-k}.$$

So we need to estimate $\#A_k$, where $A_k = \{n : F(n) = k\}$. For $n \in \mathbb{Z}$, $n \in A_k$ if and only if $\frac{2^k}{k} \leq n < \frac{2^{k+1}}{k+1}$. Then $\#A_k \approx \frac{2^{k+1}}{k+1} - \frac{2^k}{k}$, since the number of integers in an interval is within 2 of the length of the interval. It follows that $\sum_{k=2}^\infty \#\{n : F(n) = k\}2^{-k} \approx \sum_{k=2}^\infty \frac{2}{k+1} - \frac{1}{k}$. If this converged, then we could rearrange it to $\sum_{k=2}^\infty \frac{1}{k+1} - \frac{1}{k} + \sum_{k=2}^\infty \frac{1}{k+1}$. The former series is a telescoping sum converging to 1 and the latter diverges, but the sum of a convergent and a divergent sum cannot converge, so $\sum_{k=2}^{\infty} \frac{2}{k+1} - \frac{1}{k}$ cannot converge and hence neither can $\sum_{k=2}^{\infty} \#\{n : F(n) = k\}2^{-k}$, nor $\sum_{n=2}^{\infty} 2^{-F(n)}$.

**Exercise 6.** *Extra 721 Problem:* For $a_n, b_n$ sequence in $\ell^2(\mathbb{N})$, prove that

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_n b_k}{n+k} \leq C||a||_2||b||_2.$$

**Solution 6.** For $m \in \mathbb{N}$, let $I_m = \{n \in \mathbb{N} : 2^{m-1} \leq n \leq 2^m\}$. Write

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_n b_k}{n+k} = \sum_{\substack{m_1,m_2 \in \mathbb{N}\times\mathbb{N} \\ m_1 \geq m_2}} \sum_{(n,k)\in I_{m_1}\times I_{m_2}} \frac{a_n b_k}{n+k} + \sum_{\substack{m_1,m_2 \in \mathbb{N}\times\mathbb{N} \\ m_1 < m_2}} \sum_{(n,k)\in I_{m_1}\times I_{m_2}} \frac{a_n b_k}{n+k}.$$

We will prove $\sum_{\substack{m_1,m_2 \in \mathbb{N}\times\mathbb{N} \\ m_1 \geq m_2}} \sum_{(n,k)\in I_{m_1}\times I_{m_2}} \frac{a_n b_k}{n+k} \leq C||a_n||_{\ell^2}||b_n||_{\ell^2}$, the other sum will follow similarly. If $m_1 \geq m_2$, for $(n,k) \in I_{m_1} \times I_{m_2}$, $\frac{1}{n+k} \approx 2^{-m_1}$. It follows by Fubini's theorem $\sum_{(n,k)\in I_{m_1}\times I_{m_2}} \frac{a_n b_k}{n+k} \leq 2^{-m_1}||a_n||_{\ell^1(I_{m_1})}||b_n||_{\ell^1(I_{m_2})}$. By Hölder's inequality and the fact that $\#I_m = 2^m$, we have

$$2^{-m_1}||a_n||_{\ell^1(I_{m_1})}||b_n||_{\ell^1(I_{m_2})} \leq 2^{-(m_1-m_2)/2}||a_n||_{\ell^2(I_{m_1})}||b_n||_{\ell^2(I_{m_2})}.$$

Using Hölder's inequality and Fubini-Tonelli, we see that

$$\begin{aligned}
\sum_{\substack{m_1,m_2 \in \mathbb{N}\times\mathbb{N} \\ m_1 \geq m_2}} \sum_{(n,k)\in I_{m_1}\times I_{m_2}} \frac{a_n b_k}{n+k} &\leq \sum_{m_1 \geq m_2} 2^{-(m_1-m_2)/2}||a_n||_{\ell^2(I_{m_1})}||b_n||_{\ell^2(I_{m_2})} \\
&\leq \left(\sum_{m_1 \geq m_2} 2^{-(m_1-m_2)/2}||a_n||^2_{\ell^2(I_{m_1})}\right)^{1/2} \left(\sum_{m_1 \geq m_2} 2^{-(m_1-m_2)/2}||b_n||^2_{\ell^2(I_{m_2})}\right)^{1/2} \\
&\leq C||a_n||_{\ell^2(\mathbb{N})}||b_n||_{\ell^2(\mathbb{N})}
\end{aligned}$$

To see the final inequality, we sum first in the variable which is only in the power of 2. No matter than value of the other variable, the sum is always bounded above by $c = \sum_{n=1}^{\infty} \sqrt{2}^{-n} < \infty$. What is left is then bounded above by $c^{1/2}\left(\sum_{m_1\in\mathbb{N}} ||a_n||^2_{\ell^2(I_{m_1})}\right)^{1/2} = c^{1/2}||a_n||_{\ell^2(\mathbb{N})}$, as desired. This completes the problem.

**Exercise 7.** *Extra 721 Problem:* Let $x_n$ be a sequence in a Hilbert space $H$. Suppose that $x_n$ converges weakly to $x$ as $N \to \infty$. Prove that there is a subsequence $x_{n_k}$ such that

$$N^{-1}\sum_{k=1}^{N} x_{n_k}$$

converges in norm to $x$

**Solution 7.** Recall that if $x_n$ converges to $x$ weakly and $||x_n|| \to ||x||$, then $x_n \to x$, since $||x_n - x|| = \langle x_n - x, x_n - x\rangle = ||x_n||^2 + ||x||^2 - 2\langle x_n, x\rangle$. For any subsequence $x_{n_k}$, by the standard Césaro sum proof,

$$\left\langle \frac{1}{N}\sum_{k=1}^{N} x_{n_k}, h\right\rangle = \frac{1}{N}\sum_{k=1}^{N}\langle x_{n_k}, h\rangle \to \langle x, h\rangle$$

for all $h$, so it suffices to find a subsequence $x_{n_k}$ so that $\left|\left|\frac{1}{N}\sum_{k=1}^{N} x_{n_k}\right|\right|^2 \to ||x||^2$. We can expand $\left|\left|\frac{1}{N}\sum_{k=1}^{N} x_{n_k}\right|\right|^2$ to $\frac{1}{N}\sum_{k=1}^{N}\left(\frac{1}{N}\sum_{j=1}^{N}\langle x_{n_k}, x_{n_j}\rangle\right)$. We can find a subsequence $x_{n_k}$ so that for $j \geq \sqrt{k}$, $\langle x_{n_k}, x_{n_j}\rangle - \langle x_{n_k}, x\rangle \leq \frac{1}{k}$. Note that since $x_n$ converges weakly, $||x_n||^2$ is bounded by some $b$, and hence $\langle x_n, x_m\rangle \leq b^2$ for all $n, m$. Then

$$\frac{1}{N}\sum_{k=1}^{N}\left(\frac{1}{N}\sum_{j=1}^{N}\langle x_{n_k}, x_{n_j}\rangle\right) - ||x||^2 = \frac{1}{N}\sum_{k=1}^{N}\left(\frac{1}{N}\sum_{j=1}^{N}\langle x_{n_k}, x_{n_j}\rangle - \langle x_{n_k}, x\rangle\right) + \frac{1}{N}\sum_{k=1}^{N}(\langle x_{n_k}, x\rangle - \langle x, x\rangle)$$

The latter term decays to 0 since it is the Cesaro sum of a sequence that converges to 0. For the first term,

$$\left|\sum_{j=1}^{N}\langle x_{n_k}, x_{n_j}\rangle - \langle x_{n_k}, x\rangle\right| \leq \left|\sum_{j=1}^{\sqrt{k}}\langle x_{n_k}, x_{n_j}\rangle - \langle x_{n_k}, x\rangle\right| + \left|\sum_{j=\sqrt{k}+1}^{N}\langle x_{n_k}, x_{n_j}\rangle - \langle x_{n_k}, x\rangle\right| \leq \sqrt{k}b^2 + \frac{N}{k}.$$

Then

$$\left|\frac{1}{N}\sum_{k=1}^{N}\left(\frac{1}{N}\sum_{j=1}^{N}\langle x_{n_k}, x_{n_j}\rangle - \langle x_{n_k}, x\rangle\right)\right| \leq \frac{1}{N}\sum_{k=1}^{N}\frac{\sqrt{k}b^2}{N} + \frac{1}{k}.$$

As $N \to \infty$, the final sum goes to 0, completing the proof.