DAY 12 PROBLEMS AND SOLUTIONS

Exercise 1. Prove that there is a distribution $u \in \mathcal{D}'(\mathbb{R})$ so that its restriction to $(0, \infty)$ is given by

$$\langle u, f\rangle = \int_0^\infty x^{-2}\cos(x^{-2}) f(x)\, dx$$

for all $f \in C^\infty(\mathbb{R})$ compactly supported on $(0, \infty)$ and $\langle u, f\rangle = 0$ for all $f \in C^\infty(\mathbb{R})$ compactly supported on $(-\infty, 0)$.

Solution 1. One thing to try on problems like this is to use integration by parts to make the term that is bad at 0 less bad, at the cost of putting derivatives on f (which isn't really a cost at all when we are talking about distributions) and having boundary terms behave poorly at 0 (but we only care about f supported away from 0, so this won't be an issue).

In this case, this procedure works pretty efficiently, although it still took me a couple tries to get everything working. The first thing I tried was integrating x^{-2} and differentiating $\cos(x^{-2})f(x)$, which didn't immediately work because the derivatives of $\cos(x^{-2})$ created singularities at 0 as well. But doing that made me realize that $x^{-2}\cos(x^{-2})$ very nearly has an elementary antiderivative. Since $\frac{d}{dx}\sin(x^{-2}) = -2x^{-3}\cos(x^{-2})$, we will write $\int_0^\infty x^{-2}\cos(x^{-2})f(x)\, dx = \int_0^\infty x^{-3}\cos(x^{-2})(xf(x))\, dx$. Integrating by parts, this becomes $\frac{\sin(x^{-2})f(x)}{2}\Big|_\infty^0 + \int_0^\infty \frac{\sin(x^{-2})}{2}(f'(x) + xf(x))\, dx$. Since $\sin(x^{-2})$ and $x\sin(x^{-2})$ are in L^1_{loc}, the map $\langle u, f\rangle = \int_0^\infty \frac{\sin(x^{-2})}{2}(f'(x) + xf(x))\, dx$ is a well-defined distribution. It certainly sends functions supported on $(-\infty, 0)$ to 0, and undoing the integration by parts I started with, we see that equals $\int_0^\infty x^{-2}\cos(x^{-2})f(x)\, dx$ when f is compactly supported on $(0, \infty)$.

Exercise 2. On $\mathbb{R}\setminus\{0\}$ define $f(x) = |x|^{-7/2}$. Find a tempered distribution $h \in \mathcal{S}'(\mathbb{R})$ so that $f = h$ on $\mathbb{R}\setminus\{0\}$.

Solution 2. We will integrate by parts. Suppose $\varphi \in \mathcal{S}(R)$ and is compactly supported away from 0. Then integrating by parts (I'll leave it to the reader to do this carefully - you want to break up the domain into $(-\infty, 0)$ and $(0, \infty)$, and then use the fact that φ is supported away from 0 to ensure the boundary terms vanish)

$$\int |x|^{-7/2}\varphi(x)\, dx = \frac{2}{5}\int |x|^{-5/2}\varphi'(x)\, dx = \frac{4}{15}\int |x|^{-3/2}\varphi''(x)\, dx = \frac{8}{15}\int |x|^{-1/2}\varphi^{(3)}(x)\, dx.$$

The final term is a well-defined tempered distribution: it is in $L^1((-1, 1))$ and has (much better than) polynomial growth as $|x| \to \infty$. So we will take that to be h, and undoing the integration by parts written out above, we see that $h = f$ on $\mathbb{R}\setminus\{0\}$, as desired.

Exercise 3.

(1) Prove or disprove: there exists a distribution $u \in \mathcal{D}'(\mathbb{R})$ so that its restriction to $(0, \infty)$ is given by

$$\langle u, f\rangle = \int_0^\infty e^{1/x^2} f(x)\, dx$$

for all C^∞ which are compactly supported in $(0, \infty)$.

(2) Prove or disprove: there exists a distribution $u \in \mathcal{D}'(\mathbb{R})$ so that its restriction to $(0,\infty)$ is given by
$$\langle u, f\rangle = \int_0^\infty x^{-2} e^{i/x^2} f(x)\, dx$$
for all C^∞ which are compactly supported in $(0,\infty)$.

Solution 3.

(1) This is false. Suppose it was true. Then there would exist some N such that $\langle u, f\rangle \lesssim ||f||_{C^N}$ for smooth functions f supported on $[0,1]$. Let ϕ be a smooth bump function supported on $[1/4, 3/4]$, equal to 1 on $[1/3, 2/3]$, and everywhere non-negative. Define $\phi_\varepsilon = \phi(x/\varepsilon)$. Repeatedly differentiating, we see that $||\phi_\varepsilon||_{C^N} \lesssim \varepsilon^{-N}||\phi||_{C^N}$, so $\varepsilon^N\langle u, \phi_\varepsilon\rangle \lesssim 1$. Changing variables, we see that $\langle u, \phi_\varepsilon\rangle = \varepsilon\langle u_\varepsilon, \phi\rangle$, where $u_\varepsilon = e^{1/(\varepsilon x)^2}$. It suffices then to prove that for any $C > 0$, there exists ε sufficiently small such that $\langle u_\varepsilon, \phi\rangle > C\varepsilon^{-(N+1)}$. Set $\varepsilon = \frac{3}{2\log(A)^{1/2}}$ for A a large number. Then for $x \in [1/3, 2/3]$ $e^{1/(\varepsilon x)^2} \geq A$, so $\langle u_\varepsilon, \phi\rangle \geq \frac{1}{3}A$. But for A sufficiently large, $A > C(2/3\log(A))^{(N+1)/2}$, since $\lim_{A\to\infty} \frac{A}{\log(A)^{(N+1)/2}} = \infty$. Therefore, we can find an $\varepsilon > 0$ such that $\langle u_\varepsilon, \phi\rangle \gtrsim \varepsilon^{-(N+1)}$, so u cannot be a distribution.

(2) Let's write $\langle u, f\rangle = -2i\int_0^\infty \frac{-e^{i/x^2}}{2ix^3} x f(x)\, dx$. If f is compactly suported on $(0,\infty)$, then integrating parts, we see that
$$\langle u, f\rangle = 2i\int_0^\infty e^{i/x^2}[xf(x)]'\, dx = 2i\int_0^\infty xe^{i/x^2} f'(x)\, dx + 2i\int_0^\infty e^{i/x^2} f(x)\, dx.$$
As both $xe^{i/x^2}, e^{i/x^2} \in L^1_{\text{loc}}$, both are distributions, and hence so is $\langle u, f\rangle$.

Exercise 4.

(1) Suppose Λ is a distribution on $\mathbb{R}^n$ such that $\text{supp}(\Lambda) = \{0\}$. If $f \in C_c^\infty(R^n)$ satisfies $f(0) = 0$, does it follow that the product $f\Lambda = 0$ as a distribution?

(2) Suppose Λ is a distribution on $\mathbb{R}^n$ such that $\text{supp}(\Lambda) \subset K$, where $K = \{x \in \mathbb{R}^n : |x| \leq 1\}$. If $f \in C_c^\infty(\mathbb{R})$ vanishes on K, does it follow that $f\Lambda = 0$ as a distribution?

Solution 4.

(1) No. Consider $\Lambda = \partial_0'$ and $\varphi = x\psi$ for some $\psi \in C_c(\mathbb{R})$ with $\psi(0) \neq 0$. Then $\langle \Lambda, \varphi\rangle = \psi(0) \neq 0$, but $\varphi(0) = 0$.

(2) It does, although the proof is somewhat painful. I'm not sure if you could take certain steps for granted (in particular, the fact that if f vanishes to all orders on K, then $\langle \Lambda, f\rangle = 0$, which would make things much easier). As writing things carefully in Latex takes a long time, I will leave some details for you to sort out, but what I am doing is modifying the of Theorem 6.25 in Rudin's book *Functional Analysis*. You can email me if you have questions.

First, let's prove that if f vanishes on K, then $f^\alpha(x) = 0$ for any $x \in K$ and any multiindex α (since we are on $\mathbb{R}^n$, we have to worry about derivatives in different directions). We will induct on $|\alpha|$. The base case follows from our assumption that f vanishes on K. Now suppose that f that $f^\alpha(x) = 0$ for any α of order n and let β be a multi-index of order $n+1$. Suppose $f^\beta(x) = c > 0$ (the $c < 0$ case is analogous) for some $x \in K$. Since f is smooth, there is an open set U around x such that $f^\beta(y) > c/2$ for $y \in U$. Then $K \cap U$ contains an open set W on which $f^\beta(y) > c/2$.

Since $|\beta| \geq 1$, there is an index i for which $\beta_i \neq 0$. Then $f^\beta = \partial_i f^{\beta'}$ where β' is the same as β except with the ith multiindex decremented by 1. By the induction hypothesis, we know $f^{\beta'}(x) = 0$ for $x \in K$ and hence for $x \in W$, but we can find $y, y + \varepsilon e_i \in W$, in which case $|f(y) - f(y + \varepsilon e_i)| \geq \frac{c\varepsilon}{2}$, a contradiction. Hence, f^β vanishes on K.

Now suppose f vanishes on K. Then, as previously noted, f^α vanishes on K for any multiindex α. Without loss of generality, we may assume f is supported on $\overline{B}(0,2)$, as $\langle \Lambda, g\rangle = 0$ for any g compactly supported on K^c. By the definition of distributions, there exists $N \in \mathbb{N}$ such that $|\langle \Lambda, \varphi\rangle| \leq ||\varphi||_N$ for φ supported on $\overline{B}(0,2)$. Fix $\eta > 0$, we will aim to prove that $|\langle \Lambda, f\rangle| \leq \eta$. Since f vanishes to all orders at K, we know that for ε_η sufficiently small $|x| < 1 + \varepsilon_\eta$, $|D^\alpha f(x)| \leq \eta$ for all $|\alpha| = N$. It follows, by the mean-value theorem, that $|D^\beta f(x)| \leq C_\beta \eta (|x| - 1)^{|\alpha| - |\beta|}$ for fixed constants C_β. We can find a family of bump functions ψ_ε equal to 1 on $\overline{B}(0, 1+\varepsilon)$ and supported on $\overline{B}(0, 1+2\varepsilon)$ such that $|D^\alpha \psi_\varepsilon|(x) \leq C_\alpha \varepsilon^{-|\alpha|}$ for all multiindices α and $\varepsilon > 0$. Then $\psi_\varepsilon \Lambda = \Lambda$, since Λ is supported on K and $\psi_\varepsilon = 1$ on an open neighborhood of K. It follows that

$$\langle \Lambda, f\rangle = \langle \Lambda, \psi_\varepsilon f\rangle \leq \sup_{|\alpha| \leq N} \sup_{1 \leq |x| \leq 1+2\varepsilon} |D^\alpha(\psi_\varepsilon f)(x)|.$$

By the chain rule, $D^\alpha(\psi_\varepsilon f)(x) = \sum_{\beta+\gamma=\alpha} \psi_\varepsilon^\beta(x) f^\gamma(x)$, so if $2\varepsilon < \varepsilon_\eta$, we have

$$\begin{aligned} |D^\alpha(\psi_\varepsilon f)(x)| \leq \sum_{\beta+\gamma=\alpha} |D^\beta \psi_\varepsilon|(x) |D^\gamma f|(x) &\leq \sum_{\beta+\gamma=\alpha} C_\alpha C_\beta \eta (|x| - 1)^{|\alpha| - |\gamma|} \varepsilon^{-|\beta|} \\ &\leq 2 \sum_{\beta+\gamma=\alpha} C_\alpha C_\beta \eta \varepsilon^{|\alpha| - |\beta| - |\gamma|} \\ &= C\eta. \end{aligned}$$

All the constants denote fixed numbers (in particular, they do not depend on η), so since η was arbitrar,y we have that $\langle \Lambda, f\rangle = 0$, as desired.

Exercise 5. *Extra 721 Problem:*
For $f \in L^2(\mathbb{R}^+)$, define $Tf(x) = \int_0^\infty \frac{f(y)}{x+2y}\, dy$. Prove that T is a bounded operator $L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$.

Solution 5. Changing variables, we see that it suffices to prove $\tilde{T} f(x) = \int_0^\infty \frac{f(y)}{x+y}\, dy$ is bounded $L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$. Applying the principle of duality, it suffices to prove that for any $f, g \in L^2(\mathbb{R}^+)$, $\int_0^\infty \int_0^\infty \frac{f(y)g(x)}{x+y}\, dx\, dy \leq C ||f||_{L^2} ||g||_{L^2}$ for a fixed constant C. Write $I_k = [2^k, 2^{k+1}]$ for $k \in \mathbb{Z}$. Then we see that $\int_0^\infty \int_0^\infty \frac{f(y)g(x)}{x+y}\, dx\, dy = \sum_{k,j \in \mathbb{Z}} \int_{I_k} \int_{I_j} \frac{f(y)g(x)}{x+y}\, dx\, dy$. If $j \leq k$ for $(x, y) \in I_k \times I_j$ $x + y \approx 2^k$, so $\int_{I_k} \int_{I_j} \frac{f(y)g(x)}{x+y}\, dx\, dy \approx 2^{-k} ||f||_{L^1(I_k)} ||g||_{L^1(I_j)}$. By Hölder's inequality, we see that $2^{-k} ||f||_{L^1(I_k)} ||g||_{L^1(I_j)} \leq 2^{-k} 2^{j/2} 2^{k/2} ||f||_{L^2(I_k)} ||g||_{L^2(I_j)}$. Therefore, $\sum_{k \geq j} \int_{I_k} \int_{I_j} \frac{f(y)g(x)}{x+y}\, dx\, dy \approx \sum_{k \geq j} 2^{(j-k)/2} ||f||_{L^2(I_k)} ||g||_{L^2(I_j)}$.

Using Hölder's inequaltiy once more, we see that

$$\sum_{k \geq j} 2^{(j-k)/2} ||f||_{L^2(I_k)} ||g||_{L^2(I_j)} \leq \left(\sum_{k \geq j} 2^{(j-k)} ||f||^2_{L^2(I_k)}\right)^{1/2} \left(\sum_{k \geq j} 2^{(j-k)} ||g||^2_{L^2(I_j)}\right)^{1/2}.$$

Because we are summing over the range $k, j \in \mathbb{Z}, k \leq j$, fixing k and summing in j or fixing j and summing k, the 2^{j-k} always sums to 1. Therefore, we see that $\left(\sum_{k\geq j} 2^{(j-k)}||f||^2_{L^2(I_k)}\right)^{1/2} \leq ||f||_{L^2(\mathbb{R}^+)}$ and $\left(\sum_{k\geq j} 2^{(j-k)}||g||^2_{L^2(I_j)}\right)^{1/2} \leq ||g||_{L^2(\mathbb{R}^+)}$. Therefore,

$$\sum_{k\geq j} 2^{(j-k)/2}||f||_{L^2(I_k)}||g||_{L^2(I_j)} \leq ||f||_{L^2(\mathbb{R}^+)}||g||_{L^2(\mathbb{R}^+)}.$$

We handle the $j \geq k$ sum similarly and arrive at the same conclusion, completing the problem.

Exercise 6. *Extra 721 Problem:*

Let $U = \{x \in \mathbb{R}^n : |x| < 1\}$ be the open unit ball in $\mathbb{R}^n$. Let $\rho : U \to \mathbb{R}$ be a smooth function such that $\rho(0) = 0, \nabla\rho(0) \neq 0$. Let $\Sigma = \{x \in U \mid \rho(x) = 0\}$. For $x \in U$, let $d(x) = \inf_{y\in\Sigma} |x - y|$.

(1) For $x \in V = \{x \in \mathbb{R}^n : |x| \leq 1/2\}$, prove that there is a point $y \in \Sigma$ such that $d(x) = |x - y|$.
(2) For $x \in V \setminus \Sigma$ and for any $y \in \Sigma$ such that $d(x) = |x - y|$, prove that the vector $\nabla\rho(y)$ is a scalar multiple of $x - y$.
(3) Prove that there is an open set W with $0 \in W \subset V$ and a C^∞ function $\varphi : W \to \mathbb{R}$ such that for all $x \in W$, $|\varphi(x)| = d(x)$.

AN: This is a pretty old qual problem and part 2 and 3 feel more geometric (i.e. closer to a 761 problem) than most analysis qual problems now. Both require the implicit/inverse function theorem, but no theory beyond that.

Solution 6.

(1) Since $0 \in \Sigma$, $d(x) \leq \frac{1}{2}$ for all $x \in V$, and if $d(x) = |x|$, then we can take $y = 0$. Now suppose $d(x) < |x|$. Choose η such that $2d(x) < \eta < 1$ and $y_n \in \Sigma$ such that $\lim_{n\to\infty} |x - y_n| = d(x)$. For n sufficiently large, $|x - y_n| - d(x) \leq \eta - 2d(x)$, so $|x - y_n| \leq \eta - d(x)$. It follows that for n sufficiently large, $y_n \in \overline{B_{\eta-d(x)}(x)} \cap \Sigma$. This is the intersection of a compact set and a closed set and hence is itself closed. Then we can find a subsequence $y_{n_k} \to y \in \Sigma$, and since $\lim_{k\to\infty} |x - y_{n_k}| = d(x)$, we know $|x - y| = d(x)$.

(2) We would like to prove that if $y \in \Sigma$ minimizes $|x-y|^2$, then $\nabla\rho(y)$ is a scalar multiple of $x - y$. Since y is a minimizer, we know that for any smooth path $\psi : (-\varepsilon, \varepsilon) \to \Sigma$ such that $\psi(0) = y$, $\frac{d}{dz}|x - \psi(z)|^2|_{z=0} = 0$. Since $|x - \psi(z)|^2 = x^2 + \psi(z)^2 - 2x \cdot \psi(z)$, we see that $\psi'(0) \cdot (x - \psi(0)) = \psi'(0) \cdot (x - y) = 0$. We will prove that for any v orthogonal to $\nabla\rho(y)$, we can find a smooth path ψ such that $\psi'(0) = v$. This would complete the problem, since if $v \cdot (x - y) = 0$ for any v orthogonal to $\nabla\rho(y)$, we must have that $x - y$ is colinear with $\nabla\rho(y)$.

For any curve $\psi : (-\varepsilon, \varepsilon) \to \Sigma$ such that $\psi(0) = y$, $\frac{d}{dz}\rho \circ \psi = 0$, since $\rho \circ \psi \equiv 0$. On the other hand, $\frac{d}{dz}\rho \circ \psi(0) = \nabla\rho(y) \cdot \psi'(0)$, so $\psi'(0)$ is orthogonal to $\nabla\rho(y)$. It suffices then to find $n - 1$ functions $\psi_1, \ldots, \psi_n(0)$ such that $\psi_1'(0), \ldots, \psi_{n-1}'(0)$ are linearly independent.

For this, we will use the implict function theorem. We can assume $\nabla\rho(y) \neq 0$, since otherwise the $x - y$ is certainly colinear. Without loss of generality, assume $\frac{\partial}{\partial x_n}\rho(y) \neq 0$. Denote the first $n - 1$ coordinates of y as y'. Then there is a ball

$U = B(y', \varepsilon)$ and a smooth function $g : U \to \mathbb{R}$ so that $(z, g(z)) \in \Sigma$ for all $z \in U$. Define $\psi_i(\delta) = (\delta e_i + y', g(\delta e_i + y'))$ for $\delta \in (-\varepsilon, \varepsilon)$. Then $\psi_i'(0) = \left(e_i, \frac{\partial}{\partial e_i} g(y')\right)$. These are necessarily linearly independent if $e_i \neq e_j$, so as previously discussed, we are done.

(3) Use the implicit function theorem as described in the previous problem to find a neighborhood $B(0, \varepsilon)$ of 0 and a smooth function $g : B^{n-1}(0, \varepsilon) \to \mathbb{R}$ so that $(z, g(z)) \in \Sigma$ and $\nabla\rho(z, g(z)) \neq 0$. Now let $F : B^{n-1}(0, \varepsilon) \times \mathbb{R} \to V$ by $F(z, \alpha) = (z, g(z)) + \alpha \frac{\nabla\rho(z,g(z))}{|\nabla\rho(z,g(z))|}$. Using the inverse function theorem at $(0, 0)$ (I'll leave it to you to check the invertibility of $DF(0)$), we can find a neighborhood $W \subset B(0, \varepsilon)$ of 0 and a function $\Phi : W \to B^{n-1}(0, \varepsilon) \times \mathbb{R}$ such that $F \circ \Phi(y) = y$ for all $z \in W$. Then for $x \in W$, let $\varphi(x) = \Phi_n(x)$, where Φ_n denotes the n entry of Φ. Let's prove that $|\varphi(x)| = d(x)$. For each $x \in W$, we can find $y \in \Sigma$ such that $|x - y| = d(x)$. By the previous problem, it follows that there exists α such that $\alpha'\nabla\rho(y) = (x - y)$. Then $x = y + \alpha'\nabla\rho(y) = \alpha'|\nabla\rho(y)|\frac{\nabla\rho(y)}{|\nabla\rho(y)|}$. Denote $\alpha = \alpha'|\nabla\rho(y)|$. Since $y \in \Sigma \cap B(0, \varepsilon)$, $y = (z, g(z))$ for some $z \in B^{n-1}(0, \varepsilon)$. Then $x = F(z, \alpha)$. Then $\Phi(x) = (z, \alpha)$, so $\varphi(x) = \alpha$. But $d(x) = |x - y| = |\alpha'\nabla\rho(y)| = |\alpha| = |\varphi|(x)$, as desired.

Exercise 7. *Extra 721 Problem:*

Consider a differentiable function $f : \mathbb{R} \to \mathbb{R}$.

(1) Suppose the second derivative of f exists at x_0 (but not necessarily anywhere else). Show that $\lim_{h\to 0} \frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h^2} = f''(x_0)$.

(2) Suppose $\lim_{h\to 0} \frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h^2}$ exists. Recall that we have define f to be a differentiable function. Is it true that the second derivative of f exists at x_0?

Solution 7.

(1) We know that $f''(x_0) = \lim_{h\to 0} \frac{f'(x_0+h)-f'(x_0)}{h}$, so $f''(x_0) = \lim_{h\to 0} \frac{f'(x_0+h)-f'(x_0-h)}{2h}$. By L'Hopital's rule (which only requires differentiability and the numerator and denominator both going to zero), we know that

$$\lim_{h\to 0} \frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2} = \lim_{h\to 0} \frac{f'(x_0 + h) - f'(x_0 - h)}{2h}$$

which have already seen to be equal to $f''(x_0)$.

(2) No, consider, for example, $f(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$. This is differentiable every where with derivative $f'(x) = 2|x|$, so it is not twice differentiable at zero. On the other hand, $f(h) - f(-h) - 2f(0) = 0$ for all h, since f is odd, and therefore

$$\lim_{h\to 0} \frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2} = 0.$$