DAY 11 PROBLEMS AND SOLUTIONS

Exercise 1. Let $X = \{P : \mathbb{R} \to \mathbb{R} | P \text{ is a polynomial}\}$. Prove that there does not exist a norm $||\cdot||$ on X such that $(X, ||\cdot||)$ is a Banach space.

Solution 1. Suppose such a norm $||\cdot||$ exists. Let $||P||_{\text{coeff}}$ be the sup-norm of the coefficients of P. Let $K_{n,m} = \overline{\{P \in X : \deg(P) \leq n, ||P||_{\text{coeff}} \leq m\}}$. Since $\bigcup_{n,m} K_{n,m} = X$, by the Baire category theorem, some $K_{n,m}$ must have non-empty interior. Then in particular, there exists some open ball $B(P, r) \subset K_{n,m}$. This contains an element of degree $\geq n+1$ given by $Q = P + \sigma x^{n+1}$ for σ sufficiently small. Then we can write $Q = \lim_{k\to\infty} P_k$ for elements $P_k \in K_{n,m}$. Write $P_k = \sum_{j=0}^n a_{j,k} x^j$. Since each $a_{j,k}$ falls in a compact set, we may pass to a subsequence so that $a_{j,k} \to a_j$ for each j. Let $R(x) = \sum_{j=0}^n a_j x^j$. Then $P_k \to R$. But $R \neq Q$, since $\deg(R) \leq n$ and $\deg(Q) \geq n+1$, a contradiction.

Exercise 2. For $f, g \in L^2[0,1]$, let $\langle f, g\rangle = \int_0^1 f(x)\overline{g}(x)\, dx$ and set

$$g_n(x) = \frac{n^{2/3}\sin(n/x)}{xn+1}.$$

Does there exists $\alpha > 0$ such that

$$\sum_{n=1}^{\infty} |\langle f, g_n\rangle|^\alpha < \infty.$$

hold for every $f \in L^2$? *Hint: is $||g_n||_{L^2}$ a bounded sequence?*

Solution 2. The first thing you should think to do is answer the question asked in the hint. The answer it that $||g_n||_{L^2}$ is not bounded. It suffices to prove that $||g_n||_{L^2}^2 = \int_0^1 \frac{n^{4/3}|\sin(n/x)|^2}{(nx+1)^2}\, dx$ is unbounded. The $n^{4/3}$ is an obvious source of growth for this integral. On the other hand, we should expect that the $|\sin(n/x)|^2$ term will contribute less as n gets larger, essentially because the periods of $\sin(n/x)$ get narrower as n get's larger. We need to quantify the rate of decay to ensure that it is slower than the rate of growth, and hence see that $||g_n||_{L^2}^2$ is unbounded.

It suffices to prove that $\int_0^1 \frac{|\sin(n/x)|^2}{(nx+1)^2}\, dx \geq \frac{C}{n}$ for some fixed constant C. Keeping track of constants in this would be a big pain though, so I'll use asymptotic notation instead. To make the denominator go away, we will limit the domain of the integral to $[0, 1/n]$, on when $(nx+1)^2 \in [1, 4]$. We therefore reduce our problem to proving $\int_0^{1/n} |\sin(n/x)|^2\, dx \gtrsim \frac{1}{n}$. Now substitute $u = n/x$. Then $-\frac{n}{u^2}\, du = dx$, so

$$\int_0^{1/n} |\sin(n/x)|^2\, dx = n \int_{n^2}^{\infty} \frac{|\sin(u)|^2}{u^2}\, du.$$

We can bound this below by restricting the domain of integration to where $\sin(u) \geq \frac{1}{2}$, which is contained in $I = \bigcup_{k\in\mathbb{Z}\cap[n^2,\infty)} I_k$, where $I_k = [k\pi - \pi/3, k\pi + \pi/3]$. Then

$$n\int_{n^2}^{\infty} \frac{|\sin(u)|^2}{u^2}\, du \gtrsim n \sum_{k\in\mathbb{Z}\cap[n^2,\infty)} \int_{I_k} \frac{1}{u^2}\, du.$$

On I_k, $\frac{1}{u^2} \approx \frac{1}{k^2}$. Each I_k has constant length, so

$$n\sum_{k=n^2}^{\infty} \int_{I_k} \frac{1}{u^2}\, du \gtrsim n\sum_{k=n^2}^{\infty} \frac{1}{k^2}.$$

By the integral comparison test, $\sum_{k=n^2}^{\infty} \frac{1}{k^2} \approx \frac{1}{n^2}$, so $n\sum_{k=n^2}^{\infty} \frac{1}{k^2} \approx \frac{1}{n}$. Putting this all together, we see that $\int_0^1 \frac{|\sin(n/x)|^2}{(nx+1)^2}\, dx \gtrsim \frac{1}{n}$, as desired. Hence, $||g_n||_{L^2}^2$ is unbounded.

Now that we are done sorting out the hint, we should figure out why the author added the hint. In other words, what is the relation between $||g_n||_{L^2}$ being unbounded and $\sum_{n=1}^{\infty} |\langle f, g_n\rangle|^\alpha$ being infinite. It is reasonable to guess that if $||g_n||_{L^2}$ is not small, then $\sum_{n=1}^{\infty} |\langle f, g_n\rangle|^\alpha$ is not small (that is, finite) either. Turning that into an actual proof requires a bit of functional analysis.

First, note that if there exists $\alpha > 0$ such that $\sum_{n=1}^{\infty} |\langle f, g_n\rangle|^\alpha < \infty$, then by the divergence test (who knew 221 content was important for the qual!), $\lim_{n\to\infty} |\langle f, g_n\rangle|^\alpha = 0$. It follows that $\lim_{n\to\infty} |\langle f, g_n\rangle| = 0$ and, since $f \in L^2$ in arbitrary, we see that g_n converges weakly 0 in L^2. But by the open mapping theorem, we know weakly convergent sequences must be bounded in norm, and hence $||g_n||_{L^2}$ would be bounded, contradicting our earlier proof otherwise. Hence, $\sum_{n=1}^{\infty} |\langle f, g_n\rangle|^\alpha = \infty$ for any $\alpha > 0$.

Exercise 3. Let f_n be a sequence of continuous functions on $I = [0, 1]$. Suppose that for every $x \in I$ there exists an $M(x) < \infty$ so that $|f_n(x)| \leq M(x)$ for all $n \in \mathbb{N}$. Show then that $\{f_n\}$ is uniformly bounded on some interval, that is there exists $M \in \mathbb{R}$ and an interval $(a, b) \subset I$ so that $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and $x \in (a, b)$.

Solution 3. This is a Baire category theorem problem. I don't have any helpful advice on seeing that that is the right tool to use, but hopefully it will come with practice.

On to the solution. Suppose the desired conclusion does not hold, we will attempt to prove that there is a point x where $|f_n(x)|$ is unbounded. If we want to use Baire's theorem to prove something exists, we should prove that it is contained in a countable intersection of open dense sets. Let $A_M = \{x \in I : |f_n(x)| > M \text{ for some } n\}$. Since each f_n is continuous, A_M the union of the open sets $f_n^{-1}((-\infty, -M) \cup (M, \infty))$ over $n \in \mathbb{N}$ and hence is itself open. Moreover, each A_M is dense, since we have assumed that any interval contains a point x where $|f_n(x)| > M$ for some n. Then $A := \bigcap_{M\in\mathbb{N}} A_M$ is non-empty, by Baire's theorem. But for any $x \in A$, for any $M \in \mathbb{N}$, $x \in A_M$, so there exists $n \in \mathbb{N}$ such that $|f_n(x)| \geq M$. Then $M(x) \geq M$ for all M, contradicting our assumption that $M(x) < \infty$. Hence, the desired conclusion must hold..

Exercise 4. Assume that X is a compact metric space and $T : X \to X$ is a continuous map. Let $\mathcal{M}_1(T)$ denote the set of Borel probability measures on X such that $T_*\mu = \mu$. Prove:

(1) $\mathcal{M}_1(T) \neq \emptyset$.

(2) If $\mathcal{M}_1(T) = \{\mu\}$ consists of a single measure μ, then

$$\int_X f \; d\mu = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n(x)$$

for every continuous function $f : X \to \mathbb{R}$ and point $x \in X$.

Solution 4.

(1) For any $x \in X$, denote by δ_x the unit mass at x, that is, the measure satisfying $\int f(y) \; d\delta_x(y) = f(x)$. Now fix a point $x \in X$, let m_n be the Borel measure $\delta_{T^n(x)}$ for $n = 0, 1, \dots$, and for $N = 0, 1, \dots$, let $\mu_N = \frac{1}{N}\sum_{n=0}^{N-1} m_n$. Then since each m_n is a probability measure, $||\mu_N|| = \frac{1}{N}\sum_{n=0}^{N-1} ||m_n|| = 1$, so μ_N is also a probability measure. Then by Banach-Alaoglu it has a weak-$*$ convergent subsequence. Call the limit μ. Since X is compact, $1 \in C_0(X)$, so $\int 1 \; d\mu = \lim_{k\to\infty} \int 1 \; d\mu_{N_k} = 1$. We also know that μ is a positive measure, since if $f \in C(X)$ is non-negative, then $\int f \; d\mu = \lim_{k\to\infty} \int f \; d\mu_{N_k} \geq 0$, so since μ is a positive measure with unit mass, it is a probability measure. Finally, for any $f \in C(X)$, we have

$$\begin{aligned}
\left| \int f \circ T \; d\mu - \int f \; d\mu \right| &= \lim_{k\to\infty} \left| \int f \circ T - f \; d\mu_{N_k} \right| \\
&= \lim_{k\to\infty} \frac{1}{N_k} \left| \sum_{n=0}^{N_k-1} f \circ T^{n+1}(x) - \sum_{n=0}^{N_k-1} f \circ T^n(x) \right| \\
&= \lim_{k\to\infty} \frac{|f(T^{n+1}(x)) - f(x)|}{N_k} \\
&= 0.
\end{aligned}$$

Therefore $T_*\mu = \mu$, so $\mu \in \mathcal{M}_1(T)$.

(2) Denote the measure constructed in the previous part is μ_x, where x is the point we started at. If $\mathcal{M}_1(T)$ consists of a single measure μ, then $\mu = \mu_x$ for all $x \in X$. It therefore suffices to prove that $\int_X f \; d\mu_x = \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f \circ T^n(x)$ for all $f \in C(X)$. Fix $f \in C(X)$. By construction, $\int_X f \; d\mu_x = \lim_{k\to\infty} \frac{1}{N_k}\sum_{n=0}^{N_k-1} f \; \circ T^n(x)$ for some sequence $N_k \to \infty$. Now let's prove that $\int f \; d\mu = \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f \; \circ T^n(x)$. Keeping with the notation of first part, note that for any subsequence $N_j \to \infty$, $\frac{1}{N_j}\sum_{n=0}^{N_j-1} f \; \circ T^n(x) = \int f \; d\mu_{N_j}$. Following the proof in part 1, we μ_{N_j} has a subsequence $\mu_{N_{j_k}} \to \mu' \in \mathcal{M}_1(T)$, so $\lim_{k\to\infty} \frac{1}{N_{j_k}}\sum_{n=0}^{N_{j_k}-1} f \; \circ T^n(x) = \int f \; d\mu'$. But since $\mathcal{M}_1(T)$ consists of a single element, $\mu' = \mu$, so $\int f \; d\mu' = \int f \; d\mu$. Therefore, any subsequence of $\frac{1}{N}\sum_{n=0}^{N-1} f \; \circ T^n(x)$ has a subsubsequence converging to $\int f \; d\mu$, so $\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f \; \circ T^n(x) = \int f \; d\mu$, as desired.

Exercise 5. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of elements in a Hilbert space H. Suppose that $x_n \to x \in H$ weakly in H and that $||x_n|| \to ||x||$ as $n \to \infty$. Show that then $||x_n - x|| \to 0$. Would the same be true for an arbitrary Banach space in place of H?

Solution 5. We can write $||x_n - x||^2 = \langle x_n - x, x_n - x\rangle$. Using linearity of the inner product, we see that $\langle x_n - x, x_n - x\rangle = ||x_n||^2 + ||x||^2 - 2\langle x_n, x\rangle$. Since $x_n \to x$ weakly, we know $\langle x_n, x\rangle \to ||x||^2$, so since $||x_n||^2 \to ||x||^2$ as well, we have $||x_n||^2 + ||x||^2 - 2\langle x_n, x\rangle \to 0$. Hence, $||x_n - x||^2 \to 0$, so $||x_n - x|| \to 0$ as well.

The same is not true for arbitrary Banach spaces. Denote by 1 the sequence $(1, 1, 1, \dots) \in \ell^\infty$. Define the sequence $x^n = 1 - e^n \in \ell^\infty(\mathbb{N})$ (that is, $(x^n)_m = 1$ for $m \neq n$ and 0 for $m = n$). Clearly, $||x^n||_{\ell^\infty} = 1$ for all n. We also have that $||1 - x^n||_{\ell^\infty} = 1$ for all n. We will prove that x^n converges weakly to 1. It suffices to prove that e^n converges weakly to 0. Suppose otherwise. Then there exists $f \in (\ell^\infty(\mathbb{N}))^*$ such that $f(e^n) \not\to 0$. Then for some $\varepsilon > 0$, there exists a subsequence $n_k \to \infty$ such that $|f(e^{n_k})| \geq \varepsilon$ for all k. Define $\lambda \in \ell^\infty$ to be $f(e^{n_k})$ at the n_kth entry for all k and 0 everywhere else. Then $|\lambda_n| \leq ||f||$ for all n, so $\lambda \in \ell^\infty$, but $f(\lambda) = \sum_{k=1}^\infty f(e^{n_k})^2 \geq \sum_{k=1}^\infty \varepsilon^2 = \infty$, a contradiction. Hence, e^n converges weakly to 0, completing the problem.

The second part is very tricky, I think it would quite challenging to come up with a counterexample on the qual if you did not already know it.

Exercise 6. A Hamel basis for a vector space X is a collection $\mathcal{H} \subset X$ of vectors such that each $x \in X$ can be written uniquely as a finite linear combination of elements in $\mathcal{H}$. Prove that an infinite dimensional Banach space cannot have a countable Hamel basis. *Hint: Otherwise the Banach space would be first category in itself.*

Solution 6. Suppose X has a countable Hamel basis $\mathcal{H} = x_1, x_2, \dots$. Define $E_n = \text{span}\{x_1, \dots, x_n\}$.. Let's prove that E_n is closed and nowhere dense. To see it is nowhere dense, let $B(x, \varepsilon)$ be an open ball in X. If $B(x, \varepsilon) \subset E_n$, then in particular, $x \in E$, so $B(0, \varepsilon) = B(x, \varepsilon) - x \subset E_n$. But if a vector space contains an open ball at the origin, then since vector spaces are closed under scaling, it contains any open ball centered at the origin, and hence $E_n \supset X$. But this would imply X is finite dimensional, a contradiction. Therefore, we conclude that E_n must be nowhere dense.

Let's now prove that E_n is closed. You could probably use the fact that finite dimensional subspaces are closed without proof, but it is not hard to prove, so I will do so. Let $\alpha^m = \sum_{i=1}^n \alpha_i^m x_i$ define a convergent sequence in E_n with limit α. Then it is a Cauchy sequence in X, and hence a Cauchy sequence in E_n (where we equip E_n with the same norm as X). But any finite dimensional vector space over $\mathbb{R}$ is complete, so $\alpha^m \to \beta \in E_n$. Since limits in X are unique, it follows that $\beta = \alpha \in E_n$, and hence E_n is closed.

Now, the Baire category theorem tells us that $\bigcup_{n\in\mathbb{N}} E_n$ is nowhere dense, and hence $\bigcup_{n\in\mathbb{N}} E_n \neq X$. Any linear combination of a finite subset $x_{a_1}, x_{a_2}, \dots, x_{a_k} \in \mathcal{H}$ is contained in $E_{\max\{a_1,\dots,a_k\}}$, so the set of finite linear combinations of elements of $\mathcal{H}$ is a subset of E and hence is not all of X either.

Exercise 7. Show that $\ell^1(\mathbb{N}) \subsetneq (\ell^\infty)^*(\mathbb{N})$. *Hint*: Consider the sequence of averages

$$\phi_n(x) = \frac{1}{n}\sum_{j=1}^n x_j, \quad x = (x_1, x_2, \dots) \in \ell^\infty(\mathbb{N}).$$

Show that $\phi_n \in (\ell^\infty(\mathbb{N}))^*$ and consider its weak-* limit points.

Solution 7. Since ϕ_n is a linear combination of the entries of x, it is linear. To see that it is bounded, note that for any $x \in \ell^\infty$,

$$|\phi_n(x)| \leq \frac{1}{n}\sum_{j=1}^n |x_j| \leq \frac{1}{n}\sum_{j=1}^n ||x||_{\ell^\infty} = ||x||_{\ell^\infty}.$$

By Banach-Alaoglu, it has weak-$*$ limit points. Let ϕ be one of them. We now have to find some way of proving that $\phi \notin \ell^1(\mathbb{N})$. Suppose $\phi \in \ell^1(\mathbb{N})$. Let $e^n \in \ell^\infty(\mathbb{N})$ be 1 at the nth entry and 0 elsewhere. Let $E^n = \sum_{k=1}^n e^n$. Then $\phi(E^n) = \sum_{k=1}^n \phi_k$ (we've assumed that $\phi \in \ell^1(\mathbb{N})$, so this makes sense). Since ϕ is the weak limit of ϕ_{n_k}, $\phi(E^n) = \lim_{k\to\infty} \frac{n}{n_k} = 0$ for any n, so $\sum_{k=1}^n \phi_k = 0$ for any n, and hence $\phi_n = 0$ for any n, so $\phi = 0$. Therefore, $\phi(1) = 0$, but since $\phi_n(1) = 1$ for all n, this is a contradiction. Therefore, $\phi \notin \ell^1(\mathbb{N})$, so $\ell^1(\mathbb{N}) \subsetneq (\ell^\infty)^*(\mathbb{N})$, as desired.

Exercise 8. *Extra 721 Problem:*

(1) Construct a set E such that on any interval non-empty finite interval I, $0 < |E \cap I| < |I|$.
(2) Prove or give a counterexample: there exists $\alpha \in (0,1)$ and a measureable set E such that $\alpha|I| < |E \cap I| < |I|$ for every non-empty finite interval.

Solution 8.

(1) Enumerate a countable dense set in $\mathbb{R}$ (say, $\mathbb{Q}$) as $\{q_n : n \in \mathbb{N}\}$. We will construct our set E as follows: for each n, add $[q_n, q_n + \varepsilon_n]$ to the set and remove $[q_n - \varepsilon_n, q_n]$ from the set, where ε_n is a sequence satisfying $\varepsilon_n > 2\sum_{k>n} \varepsilon_k$. Then for any interval I, let I' be the middle third of I. Then there exists some m where $\varepsilon_m < |I'|$ and $q_m \in I'$. Then $E \cap I$ includes a positive measure subset of $[q_m, q_m + \varepsilon_m]$, since we only remove at most half the measure of that set in E after adding it to E, so $|E \cap I| > 0$. On the other hand, $E^c \cap I$ includes a positive measure subset of $[q_m - \varepsilon_m, q_m]$, since we only added half the measure of that set to E after removing it from E, so $|E \cap I| < |I|$.
(2) No. If there was such a set, then $f_r(x) = \frac{|E\cap[x-r.x+r]|}{2r} > \alpha$ for all x, r. Then $f(x) = \lim_{r\to 0} f_r(x) \geq \alpha$ for all x. By the Lebesgue Density theorem, it follows that $f(x) = 1$ almost everywhere, which implies that $|E \cap I| = |I|$ for all intervals I, a contradiction.

Exercise 9. *Extra 721 Problem:* Take a continuous function $K : [0,1]^2 \to \mathbb{R}$ and suppose $g \in C([0,1])$. Show that there exists a unique function $f \in C([0,1])$ such that

$$f(x) = g(x) + \int_0^x f(y)K(x,y)\ dy.$$

Solution 9. First, suppose $K < 1$. Define the operator $T : C([0,1]) \to C([0,1])$ by $T(f)(x) = g(x) + \int_0^x f(y)K(x,y)\ dy$. Then $||T(f) - T(h)||_{\sup} \leq \int_0^1 |f-h|(y)|K|(x,y)\ dy < ||f-h||_{\sup}$, so T is a contraction. By the contraction mapping theorem, T has a fixed point f_0 satisfying the desired relation. Since T is a contraction, the solution must be unique.

Suppose $|K| < M$ for some large enough M. Then

$$||T(f) - T(h)||_{C[0,(2M)^{-1}]} \leq \int_0^{(2M)^{-1}} |f-h|(y)|K|(x,y)\ dy < ||f-h||_{C[0,(2M)^{-1}]}$$

Applying the Banach contraction principle from $[0,(2M)^{-1}]$, we find a fixed point f_0 on that interval. Now suppose we have defined f_0 on some interval $[0,a]$, so that for all $x \in$

$[0,a]$, $f_0(x) = T(f_0)(x)$. Then we can extend f to $[0, a+(2M)^{-1}]$ as follows. Set $g_a(x) = g(x) + \int_0^a f_0(y)K(x,y)\,dy$ and $T_a : C([a, a+(2M)^{-1}]) \to C([a, a+(2M)^{-1}])$ by $T_a(f)(x) = g_a(x) + \int_a^{a+x} f(y)K(x,y)\,dy$. By the same reasoning as previously, we can find a fixed point f_a for T_a on $[a, a+(2M)^{-1}]$. Moreover, $f_a(a) = g_a(a) = g(a) + \int_0^a f_0(y)K(a,y)\,dy = f_0(a)$, so f_a does continuously extend f_0, so write the combined function as just f_0. Extending f_0 past a does not change the behavior of T up to a, so we still have $T(f_0)(x) = f_0(x)$ on $[0,a]$, while for $x \in [a, a+(2M)^{-1}]$, $T(f_0)(x) = g(x) + \int_0^a f_0(y)K(x,y)\,dy + \int_a^x f_0(y)K(x,y)\,dy = g(x) + \int_0^x f_0(y)K(x,y)\,dy$, as desired. After finitely many steps, we will have defined f on the entire interval $[0,1]$.

Exercise 10. *Extra 721 Problem:* Let f be a continuous real-valued function on $\mathbb{R}$ satisfying $|f(x)| \le \frac{1}{1+x^2}$. Define F on $\mathbb{R}$ by

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n)$$

(a) Prove that F is continuous and periodic with period 1.
(b) Prove that if G is continuous and periodic with period 1, then

$$\int_0^1 F(x)G(x)\,dx = \int_{-\infty}^{\infty} f(x)G(x)\,dx.$$

Solution 10.

(a) Define $F_N(x) = \sum_{n=-N}^{N} f(x+n)$. Then for any $k \in \mathbb{Z}$, $x \in [k, k+2]$, and $N = N(k)$ sufficiently large $|F_N(x) - F(x)| \le \sum_{|n|>N} \frac{1}{(x+n)^2+1} \lesssim \sum_{|n|>N} \frac{1}{n^2} \lesssim \frac{1}{N}$. It follows that F_N converges uniformly to F on $[k, k+2]$. Since each F_N is continuous on $[k,k+2]$, F is as well. Moreover, $|F(x) - F(x+1)| \le |F_N(x) - F(x)| + |F_N(x+1) - F_N(x)| + |F_N(x+1) - F(x+1)|$. By what we've already shown, the first and third terms go to 0. The second term telescopes to $|f(x+N+1) - f(x-N)| \le \frac{1}{(x+N+1)^2} + \frac{1}{(x-N)^2} \lesssim_x \frac{1}{N^2}$. It follows that $|F(x) - F(x+1)| = 0$. Since x was arbitrary, F is periodic.

(b) First, note that since G is continuous and periodic, F is continuous and periodic, and f is absolutely integrable, both integrals always exist. Using the definitions in the previous part, as well as the periodicity of G, we see that

$$\begin{aligned}
\int_{-N}^{N+1} f(x)G(x)\,dx &= \sum_{n=-N}^{N} \int_n^{n+1} f(x)G(x)\,dx \\
&= \sum_{n=-N}^{N} \int_0^1 f(x+n)G(x)\,dx \\
&= \int_0^1 \sum_{n=-N}^{N} f(x+n)G(x)\,dx \\
&= \int_0^1 F_N(x)G(x)\,dx.
\end{aligned}$$

It follows that

$$\left|\int_{-\infty}^{\infty} f(x)G(x)\ dx - \int_0^1 F(x)G(x)\ dx\right|$$
$$\leq \left|\int_{-\infty}^{\infty} f(x)G(x)\ dx - \int_{-N}^{N+1} f(x)G(x)\ dx\right| + \left|\int_0^1 F_N(x)G(x)\ dx - \int_0^1 F(x)G(x)\right|.$$

The first term goes to 0 by the integrability of $f(x)G(x)$, the second term is bounded about by $\sup_{x\in[0,1]} |F_N(x) - F(x)| \int_0^1 G(x)\ dx$, which goes to 0 by the first part.