DAY 10 PROBLEMS AND SOLUTIONS

Exercise 1. Suppose that E is a measurable set of real numbers with arbitrarily small periods (that is, there exists a sequence of real numbers $p_i \to 0$ such that $E + p_i = E$). Prove that either E or it's complement has measure 0.

Solution 1. Suppose $m(E^c) \neq 0$, let's prove $m(E) = 0$. If $m(E) > 0$, then E has a Lebesgue point x. Since $m(E^c) \neq 0$, there is an interval, $[a, a+1]$ such that $\varepsilon = m(E^c \cap [a, a+1]) > 0$. Since x is a Lebesgue point of E, there exists $\delta > 0$ such that if $\delta' < \delta$, then $\frac{m([x-\delta', x+\delta'] \cap E)}{2\delta'} > 1 - \varepsilon/2$. Choose a period p_i such that $p_i < \min(\delta, \varepsilon/12)$. Then we can cover $[a, a+1]$ with intervals $[x_1 - p_i, x_1 + p_i], [x_2 - p_i, x_2 + p_i], \ldots, [x_m - p_i, x_m + p_i]$, where $x_i = x + n_i p_i$ for some $n_i \in \mathbb{Z}$ and $[x_i - p_i, x_i + p_i] \cap [a, a+1] \neq \emptyset$. It follows that $m \geq \frac{1}{2p_i}$. Since Lebesgue measure is translation invariant,

$$m([x_1 - p_i, x_1 + p_i] \cap E) = m(([x - p_i, x + p_i] + np_i) \cap E) = m([x - p_i, x + p_i] \cap E) > 2p_i(1 - \varepsilon/2).$$

We know $2mp_i(1-\varepsilon/2) < m\left(\bigcup_{i=1}^m [x_i - p_i, x_i + p_i] \cap E\right)$ and $\bigcup_{i=1}^m [x_i - p_i, x_i + p_i] \setminus [a, a+1]$ consists of two intervals $[\alpha, a]$ and $[a+1, \beta]$ of length at most p_i, so $m\left(\bigcup_{i=1}^m [x_i - p_i, x_i + p_i] \cap E\right) - m(E \cap [a, a+1]) < 2p_i$, and hence

$$m(E \cap [a, a+1]) > 2mp_i(1 - \varepsilon/2) - 2p_i > (1 - \varepsilon/2) - 2p_i > 1 - 2\varepsilon/3.$$

It follows that $m(E^c \cap [a, a+1]) < \varepsilon/3$, contradicting our definition $m(E^c \cap [a, a+1]) = \varepsilon$.

Exercise 2. Let $\mathbf{T} \subset \mathbb{C}$ be the unit circle. We say that $G \subset \mathbf{T}$ is a subgroup of $\mathbf{T}$ if $1 \in G$, $\zeta_1, \zeta_2 \in G$ implies $\zeta_1\zeta_2 \in G$ and $\zeta_1 \in G$ implies $\zeta_1^{-1} \in G$.

(1) What are the compact subgroups of $\mathbf{T}$?
(2) Give an example of an infinite subgroup $G \subsetneq \mathbf{T}$.
(3) Prove or give a counterexample: there are no measurable subgroups $G \subsetneq \mathbf{T}$ with $|G| > 0$.

Solution 2.

(1) We know $\mathbf{T}$ is a compact subgroup, as are the nth roots of unity $F_n = \{e^{2\pi i k/n} : k = 0, 1, \ldots, n-1\}$. We will show that these are the only compact subgroups.

First, any finite subgroup $G \subset \mathbf{T}$ must be a collection F_n for some n, because it must have some element $e^{2\pi i(1/x)}$ closest to 1. If x is not an integer, let n be the least integer greater than x. Then $0 < \frac{n}{x} - 1 < \frac{1}{x}$, since otherwise would contradict the minimality of n, and therefore $e^{2\pi i n/x}$ is closer to 1 than $e^{2\pi i 1/x}$, a contradiction. Therefore, x must be an integer. If x is an integer n, then $G \supset F_n$, and if there exists some element $e^{2\pi i\theta} \in \mathbf{T} \setminus F_n$, then $\theta - k/n < 1/n$ for some choice of k, in which case $e^{2\pi i(\theta - k/n)}$ is closer to 1 than $e^{2\pi i(1/x)}$, a contradiction. Hence, $G = F_n$.

If G is infinite, then since $\mathbf{T}$ is compact, it has a limit point $\xi \in \mathbf{T}$, and since G is closed, $\xi \in G$. It follows that G contains a sequence of elements ξ_i converging to ξ, in which case $e^{2\pi i\theta_i} = \frac{\xi_i}{\xi} \in G$ is a sequence of elements converging to 1. Let's prove that G is dense in $\mathbf{T}$. For any $e^{2\pi i\eta} \in \mathbf{T}$, we can find m_i such that $|\eta - m_i\theta_i| < \theta_i$. We know that $e^{2\pi i m\theta_i} \in G$ for all $m \in \mathbb{Z}$ and $\lim_{i\to\infty} \theta_i = 0$, so $\lim_{i\to\infty} m_i\theta_i = \eta$ and hence $e^{2\pi i\eta} \in \overline{G}$. But since G is closed, we have that $e^{2\pi i\eta} \in G$, and since $e^{2\pi i\eta}$ was an arbitrary element of $\mathbf{T}$, we have that $G = \mathbf{T}$.

(2) Take $G = \{e^{2\pi i q} : q \in \mathbb{Q}\}$. This is a subgroup, because $\mathbb{Q}$ is a group and it is clearly infinite.

(3) This is true. Let G be a subgroup of $\mathbf{T}$ with $|G| > 0$. First, let's prove that G contains an interval. We could do this using Young's inequality, but I put this on the Lebesgue differentiation theorem worksheet so I guess I will use the Lebesgue differentiation theorem to do this. Let's consider G as subset of $[-\pi, \pi)$ closed under addition. Let θ be a Lebesgue point for G, and without loss of generality, we may assume that $\theta \in [0, \pi/2]$ (the set of Lebesgue points of G is closed under translation by elements of G, and the argument in part 1 tells us that G contains elements arbitrarily close to 0, so we can repeatedly translate θ by those elements until it is where we want it, it will be convenient to do given our mapping of G onto an interval). Then there exists $\delta > 0$ such that $\frac{m(G\cap[\theta-r,\theta+r])}{2r} > .99$ for all $r \leq \delta$. Suppose there exists $\eta \in [2\theta-\delta/2, 2\theta+\delta/2]\setminus G$. For any $\xi \in [\theta-\delta/2, \theta+\delta/2]$, we know one of ξ and $\eta-\xi$ cannot be in G. Now $\xi, \eta-\xi \in [\theta-\delta, \theta+\delta]$, so for all $\eta - G\cap[\theta-\delta/2, \theta+\delta/2] \subset [\theta-\delta, \theta+\delta]\setminus G$. We know that $m(G\cap[\theta-\delta/2, \theta+\delta/2]) > .99\delta$, so $m(\eta - G\cap[\theta-\delta/2, \theta+\delta/2]) > .99\delta$, and hence $m([\theta-\delta, \theta+\delta]\setminus G) > .99\delta$, so $m([\theta-\delta, \theta+\delta]\cap G) < 1.01\delta$, contradicting our assumption that $m([\theta-\delta, \theta+\delta]\cap G) > 1.98\delta$. Hence, G contains an interval. As previously discussed, G contains arbitrarily small translations, so we can translate the interval in G to cover all of $\mathbf{T}$. Hence, if G has positive measure, then it cannot be strictly contained in $\mathbf{T}$, as desired.

Exercise 3. Let $I = [0, 1)$ and given $N \in \mathbb{N}$ consider the dyadic intervals $I_{j,N} = [j2^{-N}, (j+1)2^{-N})$ for $j \in \{0, 1, \ldots, 2^N - 1\}$. For a function $f \in L^1(I)$, define a sequence of function $E_N f : I \to \mathbb{R}$ by

$$E_N f(x) = 2^N \int_{I_{j,N}} f(t)\, dt \qquad \text{for } x \in I_{j,N}.$$

Show that $\lim_{N\to\infty} E_N f(x) = f(x)$ for a.e. $x \in I$.

Solution 3. We'll just run the usual proof of the Lebesgue differentiation theorem, with minor modifications to fit this case. Let $Mf(x) = \sup_{x\in I_{j,N}} 2^N \int_{I_{j,N}} |f|(t)\, dt$. Let's first prove that M enjoys the weak-type bound $||Mf||_{L^{1,\infty}} \leq C||f||_{L^1}$. We need to show that if $E_\delta = \{x \in \mathbb{R} : Mf(x) \geq \delta\}$, then $\delta m(E_\delta) \leq C||f||_{L^1}$ for an absolute constant C (that is, one that does not depend on f or δ). Now, for all $x \in E_\delta$, we can find an interval I_{j_x,N_x} containing x such that $\int_{I_{j_x,N_x}} |f|(x)\, dx \geq 2^{-N}\delta$. This gives a cover for E_δ. By the Vitali covering theorem, we can find a finite collection of disjoint sets $I_{j_1,N_1}, \ldots, I_{j_m,N_m}$ such that if $\tilde{I}_{j,N}$ is the interval of length $\frac{5}{2^N}$ with the same center as $I_{j,N}$, then $\bigcup_{i=1}^m \tilde{I}_{j_i,N_i}$ covers E_δ, and hence $m(E_\delta) \leq 5\sum_{i=1}^m 2^{-N_i}$. Now we know

$$\int_I |f|(x)\, dx \geq \sum_{i=1}^m \int_{I_{j_i,m_i}} |f|(x)\, dx \geq \sum_{i=1}^m 2^{-N_i}\delta \geq \delta m(E_\delta)/5.$$

Since f and δ were arbitrary, we see that

$$||Mf||_{L^{1,\infty}} \leq 5||f||_{L^1},$$

as desired.

Now, let's complete the proof of the exercise. Define $E_\delta = \{x \in \mathbb{R} : \limsup_{N\to\infty} |E_N f(x) - f(y)| > \delta\}$. It suffices to show that $m(E_\delta) = 0$ for any $\delta > 0$, since then the set of points where $\lim_{N\to\infty} E_N f(x) \neq f(x)$ has measure 0. Let g be a continuous approximation of f, such that $||f - g||_{L^1} < \varepsilon$ for some small ε to be determined later. Then for any x and dyadic interval $I_{j,N}$ containing x,

$$2^N \int_{I_{j,N}} |f(t) - f(x)|\, dt \leq 2^N \int_{I_{j,N}} |f(t) - g(t)|\, dt + 2^N \int_{I_{j,N}} |g(t) - g(x)|\, dt + |g(x) - f(x)|.$$

By the bound on Mf proven in the previous paragraph, $\sup_{x\in I_{j,N}} 2^N \int_{I_{j,N}} |f(t)-g(t)|\, dt > \delta/3$ on a set of measure $< \frac{C||f-g||_{L^1}}{\delta}$. By Markov's inequality (which says $||f||_{L^{1,\infty}} \leq ||f||_{L^1}$ and follows very easily from the layer-cake formula), $|g(x) - f(x)| > \delta/3$ on a set of measure $< \frac{3||f-g||^{L^1}}{\delta}$. And since g is continuous, $\limsup_{N\to\infty} 2^N \int_{I_{j,N}} |g(t)-g(x)|\, dt = 0$. It follows that $\limsup_{N\to\infty} |E_N f(x) - f(y)| \geq \delta$ on a set of measure $< \frac{C||f-g||_{L^1}}{\delta} < \frac{C\varepsilon}{\delta}$. Taking ε arbitrarily small, we conclude that $m(E_\delta) = 0$. Since δ was arbitrary, we see that $\lim_{N\to\infty} |E_N f(x) - f(x)| = 0$ almost everywhere.

Exercise 4. For $x, y \in \mathbb{R}$, let $K(y) = \pi^{-1}(1 + y^2)^{-1}$, and for $t > 0$ let

$$P_t f(x) = \int_{-\infty}^{\infty} t^{-1} K(t^{-1}y) f(x - y)\, dy.$$

(1) Show that if f is continuous and compactly supported, then

$$\lim_{t\to 0^+} \sup_{x\in\mathbb{R}} |P_t f(x) - f(x)| = 0.$$

(2) Let $p \geq 1$. For $f \in L^p(\mathbb{R})$ denote by Mf the Hardy-Littlewood maximal function of f. Show that there is a constant $C > 0$ so that for all $f \in L^p(\mathbb{R})$ the inequality

$$|P_t f(x)| \leq CMf(x)$$

holds for every $x \in \mathbb{R}$ and every $t > 0$.

(3) If $f \in L^1(\mathbb{R})$, prove that $\lim_{t\to 0+} P_t f(x) = f(x)$ for almost every $x \in \mathbb{R}$.

Solution 4.

(1) Fix $\varepsilon > 0$, let's prove that for t sufficiently large and all $x \in \mathbb{R}$, $|P_t(x) - f(x)| < \varepsilon$. Since f is continuous and compactly supported, $M := \sup_{x\in\mathbb{R}} |f(x)| < \infty$ and there exists $\delta > 0$ such that if $|h| < \delta$, then $|f(x + h) - f(x)| < \frac{\varepsilon}{2}$. Note that $\int K(y)\, dy = \frac{\arctan(x)}{\pi} + C$, so $\int_{-\infty}^{\infty} K(y)\, dy = 1$, and by change of variables, $\int_{-\infty}^{\infty} t^{-1}K(t^{-1}y)\, dy = 1$. We also know that by dominated convergence that $\lim_{t\to 0^+} \int_{|y|\geq\delta/t} K(y)\, dy = 0$, so for t sufficiently large, $\int_{|y|\geq\delta/t} K(y)\, dy \geq \frac{\varepsilon}{2M}$.

Then for any $x \in \mathbb{R}$,

$$\begin{aligned}
|P_t f(x) - f(x)| &= \left| \int_{-\infty}^{\infty} t^{-1} K(t^{-1}y)[f(x-y) - f(x)]\, dy \right| \\
&\leq \left| \int_{-\infty}^{\infty} K(y)[f(x-ty) - f(x)]\, dy \right| \\
&\leq \int_{|y|<\delta/t} K(y)|f(x-ty) - f(x)|\, dy + \int_{|y|\geq\delta/t} K(y)|f(x-ty) - f(x)|\, dy \\
&\leq \int_{|y|<\delta/t} K(y)\frac{\varepsilon}{2}\, dy + \int_{|y|\geq\delta/t} K(y) M\, dy \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

Since $\varepsilon > 0, x \in \mathbb{R}$ were arbitrary, $\lim_{t\to 0^+} \sup_{x\in\mathbb{R}} |P_t f(x) - f(x)| = 0$.

(2) Recall that the Hardy-Littlewood maximal function is defined to be

$$Mf(x) = \sup_{r>0} \frac{1}{2r} \int_{-r}^{r} |f|(x-y)\, dy.$$

The idea here is to approximate the kernel $K(y/t)$ from above with integrals of f over intervals, then bound those from above with the Hardy-Littlewood maximal function.

For $n \in \mathbb{N}$ and $t > 0$, define $I_{n,t} = \{x : K(x/t) \geq 2^{-n}]\}$ and define $I_{0,t} = \emptyset$. Since K increases for negative x and decreases for positive x, $I_{n,t}$ are intervals, since K is bounded above by 1, $\mathbb{R} = \bigcup_{n\in\mathbb{N}} I_{n,t}$, and I_n is an increasing family: $I_1 \subset I_2 \subset \ldots$. We can compute that if $y \in I_{n,t}$, then $1 + (y/t)^2 \leq 2^n$, so $y^2 \leq t^2 2^n$, and hence $|y| \leq t2^{n/2}$, and hence $I_{n,t} \subset [-t2^{n/2}, t2^{n/2}]$. It follows that

$$\begin{aligned}
P_t f(x) &\leq \sum_{n=1}^{\infty} \frac{1}{t} \int_{I_{n,t}\setminus I_{n-1,t}} 2^{1-n} |f|(x-y)\, dy \\
&\leq \sum_{n=1}^{\infty} \frac{2}{2^n t} \int_{I_{n,t}} |f|(x-y)\, dy \\
&\leq \sum_{n=1}^{\infty} \frac{2}{2^n t} \int_{-t2^{n/2}}^{t2^{n/2}} |f|(x-y)\, dy \\
&\leq \sum_{n=1}^{\infty} \frac{4 2^{n/2} t}{2^n t} Mf(x) \\
&\leq 4\frac{\sqrt{2}}{\sqrt{2}-1} Mf(x).
\end{aligned}$$

Where the last inequality is by summing the geometric series $\sum_{n=0}^{\infty} \frac{1}{\sqrt{2}^n}$. We will take the latter constant as C, note that it does not depend on t or x, so this reasoning completes the problem.

(3) I assume we can take the weak-L^1 boundedness of the Hardy-Littlewood maximal function for granted, in which case we can conclude from the previous part that $\tilde{M}f(x) = \sup_{t>0} P_t|f|(x)$ is weak-L^1 bounded as well, that is $||Mf||_{L^{1,\infty}} \leq ||f||_{L^1}$.

From here, we will reproduce the end of the proof of the Lebesgue differentiation theorem.

Fix $m \in \mathbb{N}$ and let $E_m = \{x \in \mathbb{R} : \limsup_{t\to 0^+} |P_t f(x) - f(x)| \geq 1/m\}$. For $x \notin \bigcup_{m\in\mathbb{N}} E_m$, $\limsup_{t\to 0^+} |P_t f(x) - f(x)| = 0$, so $\lim_{t\to 0} P_t f(x) = f(x)$. We will conclude by proving that $|E_m| = 0$ for all m, so $\left|\bigcup_{m\in\mathbb{N}} E_m\right| = 0$. Fix $m \in \mathbb{N}$ and $\varepsilon > 0$, let's prove that $|E_m| < \varepsilon$. Since continuous, compactly supported functions are dense in $L^1(\mathbb{R})$, we can find $g \in C_c(\mathbb{R})$ such that $||g - f||_{L^1}$ is a small value to be determined later. It follows that g differs from f by more than $\frac{1}{3m}$ on a set of F of measure $\leq 3m||g - f||_{L^1}$. We know for all $x \in \mathbb{R}$, $|P_t f(x) - f(x)| \leq |P_t f(x) - P_t g(x)| + |P_t g(x) - g(x)| + |g(x) - f(x)|$. If $x \in F^c$, then $|g(x) - f(x)| < \frac{1}{3m}$. We know by the first problem that $\limsup_{t\to 0} |P_t g(x) - g(x)| = 0$. By definition, $|P_t f(x) - P_t g(x)| \leq \tilde{M}(f - g)(x)$, so by the weak-$L^1$ bound previously discussed, $|P_t f(x) - P_t g(x)| \leq \frac{C}{3m}$ outside of a set G of measure $\leq 3m||f - g||_{L^1}$. Then for $x \notin F \cup G$, $\limsup_{t\to 0^+} |P_t f(x) - f(x)| \leq \frac{2}{3m}$, so $x \notin E_m$. Since $E_m^c \supset (F \cup G)^c$, we have that $E_m \subset F \cup G$, so $|E_m| < |F| + |G| \leq m(3 + 3C)||f - g||_{L^1}$. Choose $||f - g||_{L^1} \leq \frac{\varepsilon}{m(3+3C)}$, and we see that $|E_m| < \varepsilon$. Since ε was arbitrary, we have $|E_m| = 0$, as desired.

Exercise 5. Given a real number x, let $\{x\}$ denote the fractional part of x. Suppose α is an irrational number and define $T : [0, 1] \to [0, 1]$ by

$$T(x) = \{x + \alpha\}.$$

Prove: If $A \subset [0, 1]$ is measurable and $T(A) = A$, then $|A| \in \{0, 1\}$.

Solution 5. If $|A| = 0$ or $|A| = 1$, we are done. So assume $|A| = c \in (0, 1)$. Then A has a Lebesgue point for χ_A, that is, a point $x \in (0, 1)$ where $\lim_{r\to 0} \frac{m([x-r,x+r]\cap A)}{2r} = 1$. Fix r_0 and sufficiently small so that $m([x - r_0, x + r_0] \cap A) > 2r_0\varepsilon$, where ε is a number to be determined later. We will use this to prove that $|A| > c$, a contradiction.

Next we will prove that T is measure preserving on all measurable sets. This is easy to check for intervals. Since $T([0, 1]) = [0, 1]$ and T is a bijection, if T is measure preserving on a set, it is also measure preserving on it's complement. Finally, if T is measure preserving on a collection of sets, it is measure preserving on their disjoint union. It follows by the $\pi - \lambda$ theorem that T is measure preserving on all measurable sets.

Since T is injective, $T(A \cap B) = T(A) \cap T(B)$ for any sets A, B. And T is measure preserving, so $m(T(A)) = m(A)$ for any set A. Then for any interval I, $m(I \cap A) = m(T(I \cap A)) = m(T(I) \cap T(A)) = m(T(I) \cap A)$. We can iterate this to prove that $m(I \cap A) = m(T^n(I) \cap A)$. Now if $I = [x - r_0, x + r_0]$ and $|T^n(x) - x| > 2r_0$, then $T^n([x - r_0, x + r_0]) \cap [x - r_0, x + r_0] = \emptyset$, as both are intervals with length r_0 and center seperated by r_0. Assume we can find $k = \lfloor \frac{1}{2r_0} \rfloor - 1$ values $0 = n_1, \ldots, n_k$ such that $|T^{n_j}(x) - T^{n_i}(x)| > 2r_0$ for all $j \neq i$. It follows that $m(A) \geq \sum_{i=1}^{m} m(T^{n_i}(I) \cap A) \geq m2r_0\varepsilon \geq 2r_0(\frac{1}{2r_0} - 2)\varepsilon \geq (1 - 4r_0)\varepsilon$. By choosing ε to be very close to 1 and then r_0 very close to 0, we can ensure $m(A) > c$, a contradiction.

We will conclude by proving our assumption that we can find k well-spread out points. It suffices to prove that $\{T^n(x) : n \in \mathbb{N}\}$ is dense for any x. The mapping $q : \mathbb{R} \to \mathbf{T}$ sending x to $e^{2\pi ix}$ is an open map, so the preimage of a dense set is dense, and since q restricts to a bijection on $[0, 1)$, $\{T^n(x) : n \in \mathbb{N}\}$ is dense in $[0, 1]$ as long as $q(\{T^n(x) : n \in \mathbb{N}\})$ is dense in $\mathbf{T}$. But $q(\{x\}) = q(x)$, so $q(T^n(x)) = q(x + n\alpha) = e^{2\pi i(x+n\alpha)}$. Therefore,

$q(\{T^n(x) : n \in \mathbb{N}\}) = \{e^{2\pi i(x+\alpha n)} : n \in \mathbb{Z}\}$. Since multiplication by $e^{2\pi ix}$ is an automorphism of $\mathbf{T}$, we may assume $x = 0$, so we need only prove that $G = \{e^{2\pi i\alpha n} : n \in \mathbb{Z}\}$ is dense in $\mathbf{T}$. But G is an infinite subgroup of $\mathbf{T}$, so following the proof in the solution to exercise 2, we see that it is dense.

Jacob Denson has a more direct solution to this problem in his notes.