DAY 1 PROBLEMS AND SOLUTIONS

Exercise 1. For a sequence (a_k) let $s_n = \sum_{k=1}^{n} a_k$ and $\sigma_L = \frac{1}{L}\sum_{n=1}^{L} s_n$. We say that $\sum_{k=1}^{\infty} a_k$ is Cesáro summable to S if $\lim_{L\to\infty} \sigma_L = S$.

(1) Prove: $s_n - \sigma_n = \frac{(n-1)a_n + (n-2)a_{n-1} + \cdots + a_2}{n}$.
(2) Prove: If $\sum_{k=1}^{\infty} a_k$ is Cesáro summable to S and if $\lim_{k\to\infty} k a_k = 0$, then $\sum_{k=1}^{\infty} a_k$ converges and $\sum_{k=1}^{\infty} a_k = S$.

Solution 1.

(1) Let's induct on n. If $n = 1$, $s_n = \sigma_n = a_1$, so the desired equality holds. Now suppose the equality holds for some n. Note that $\sigma_{n+1} = \frac{n}{n+1}\sigma_n + \frac{1}{n+1}s_{n+1}$. Then $s_{n+1} - \sigma_{n+1} = \frac{n}{n+1}(s_{n+1} - \sigma_n)$. Applying the inductive hypothesis,

$$s_{n+1} - \sigma_n = s_{n+1} - s_n + \frac{(n-1)a_n + \cdots + a_2}{n} = \frac{n a_{n+1} + (n-1)a_n + \cdots + a_2}{n}.$$

Then $\frac{n}{n+1}(s_{n+1} - \sigma_n) = \frac{n a_{n+1} + (n-1)a_n + (n-2)a_{n-1} + \cdots + a_2}{n+1}$, as desired.

(2) We need to prove that $\lim_{k\to\infty} s_k = S$, so it suffices to prove

$$\lim_{k\to\infty} s_k - \sigma_k = \lim_{k\to\infty} \frac{(k-1)a_k + (k-2)a_{k-1} + \cdots + a_2}{k} = 0.$$

Fix $\varepsilon > 0$ and choose N large enough that $|k a_k| < \varepsilon/2$, and hence $|(k-j)a_k| < \varepsilon/2$ for all $k \geq N$ and $j < k$. Then

$$\frac{(k-1)a_k + \cdots + a_2}{k} = \frac{(k-1)a_k + \cdots + (N-1)a_N}{k} + \frac{(N-2)a_{N-1} + \cdots + a_2}{k}$$

The first expression on the right is $< \varepsilon/2$. Taking $k > N$ sufficiently large makes the second expression $< \varepsilon/2$ as well, since the numerator is fixed. Then the whole sum is $< \varepsilon$, so $|s_k - \sigma_k| < \varepsilon$. Since ε was arbitrary, $\lim_{k\to\infty} s_k - \sigma_k = 0$, and we are done.

Exercise 2. Let

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < x_1^2 \leq 1/2\}.$$

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x) = (x_1^2 + x_2^2)^{-b/2} |\log(x_1^2 + x_2^2)|^{-\gamma}.$$

Determine for which values $b > 0, \gamma \in \mathbb{R}$, $\int_\Omega f(x)\, dx$ is finite.

Solution 2. Let's radially integrate. The only part of f that could cause the integral to diverge is the singularity at 0 (depending on the sign of γ, there might be a singularity where $x_1^2 + x_2^2 = 1$ as well, but we can never get very close to it because $\Omega \subset B_{3/4}(0)$), so we are free to ignore parts of the domain outside of the circle of radius 1/2. We also have that both Ω and f are symmetric about the x_2-axis, so $\int_\Omega f(x)\, dx < \infty$ if and only if $\int_{\Omega^+} f(x)\, dx > 0$, where $\Omega^+ = \{(x_1, x_2) \in \Omega : x_1 > 0\}$. We have

$$\int_{\Omega^+} f(x)\, dx = \int_0^{1/2} \frac{r^{-b}}{|\log(r)|^\gamma} M(r)\, dr,$$

where $M(r)$ denotes the Lebesgue measure of the set

$$A(r) := \{\theta \in [0, \pi/2] : (r\cos(\theta), r\sin(\theta)) \in \Omega\} = \{\theta \in [0, \pi/2) : r^2\cos^2(\theta) > r\sin(\theta) > 0\}$$

Let's prove that $M(r) \approx r$ ($a \approx b$ means $ca < b < Ca$ for positive constants c, C), justifying replacing $M(r)$ in our integral with r. Applying the Pythagorean identity, we see that $A(r) = [0, \xi)$, where $u = \sin(\xi)$ solves $ru^2 + u - r = 0$, and hence $M(r) = \xi$. Using the quadratic formula, we have $\sin(\xi) = \frac{\sqrt{1+4r^2}-1}{2r}$. Since $\sin(\xi) \le \xi \le 2\sin(\xi)$ for $\xi \in [0, \pi/2]$, we know that $M(r) \approx \frac{\sqrt{1+4r^2}-1}{2r}$. Taylor expanding $r \mapsto \sqrt{1+4r^2}$, we have $\sqrt{1+4r^2} = 1 + 2r^2 + O(r^4)$. Since $r \in [0, 1/2]$, $2r^2 + O(r^4) \approx r^2$, and hence $M(r) \approx \frac{r^2}{r} = r$, as desired.

Substituting this in, we see that the original integral converges if and only if

$$\int_0^{1/2} \frac{r^{1-b}}{|\log(r)|^\gamma}\, dr < \infty.$$

For me, it is more comfortable to first substitute $u = 1/r$. The integral becomes $\int_2^\infty \frac{u^{b-3}}{|\log(u)|^\gamma}\, dr$ and now the singularity is at ∞. Any (positive) power of u grows faster than any power of $\log(u)$, so if $b - 3 < -1$, the integral converges and if $b - 3 > -1$, it diverges. Equivalently, if $b < 2$, the integral converges and if $b > 2$ it diverges, no matter what γ is. On the other hand, if $b = 2$, then the integral becomes $\int_2^\infty \frac{1}{u|\log(u)|^\gamma}\, du$. Substituting $v = \log(u)$, this becomes $\int_{\log(2)}^\infty \frac{1}{v^\gamma}\, dv$, which is finite if and only if $\gamma > 1$.

Exercise 3. Consider the series

$$\sum_{n=1}^\infty \frac{1}{n}\sin\left(\frac{x}{n}\right)$$

.

(1) Does it converge uniformly on $[0, 1]$?
(2) Does it converge uniformly on $[0, \infty)$?

Solution 3.

(1) It does converge uniformly in the given range. It suffices to prove that for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $M \ge N$, then $\left|\sum_{n=M}^\infty \frac{1}{n}\sin\left(\frac{x}{n}\right)\right| \le \varepsilon$. Note that by the standard result that $|\sin(x)| \le |x|$,

$$\left|\sum_{n=N}^\infty \frac{1}{n}\sin\left(\frac{x}{n}\right)\right| \le \sum_{n=N}^\infty \frac{1}{n}\left|\sin\left(\frac{x}{n}\right)\right| \le \sum_{n=N}^\infty \frac{x}{n^2} \le \sum_{n=N}^\infty \frac{1}{n^2}$$

Since $\sum_{n=1}^\infty \frac{1}{n^2} < \infty$, we know that $\lim_{N\to\infty} \sum_{n=N}^\infty \frac{1}{n^2} = 0$, and hence for N sufficiently large and all $M \ge N$, $\sum_{n=M}^\infty \frac{1}{n^2} < \varepsilon$. For these values of M, it follows that $\left|\sum_{n=M}^\infty \frac{1}{n}\sin\left(\frac{x}{n}\right)\right| \le \varepsilon$ as well, so we have uniformy convergence.

(2) Suppose the series converged uniformly on $[0, \infty)$. Then there exists N sufficiently large so that for all $M \ge N$ and all x, $\left|\sum_{n=M}^\infty \frac{1}{n}\sin\left(\frac{x}{n}\right)\right| < 1/100$. It follows that $\left|\sum_{n=N}^\infty \frac{1}{n}\sin\left(\frac{N}{n}\right)\right| < 1/100$. But $\frac{N}{n} \in [0, 1]$ for all $n \ge N$, so $\sin\left(\frac{N}{n}\right) \ge \frac{N}{10n}$. It follow that $\sum_{n=N}^\infty \frac{1}{n}\sin\left(\frac{N}{n}\right) \ge \frac{N}{10}\sum_{n=N}^\infty \frac{1}{n^2}$. By the integral test, $\sum_{n=N}^\infty \frac{1}{n^2} \ge \frac{1}{N}$, so $\frac{N}{10}\sum_{n=N}^\infty \frac{1}{n^2} \ge \frac{1}{10}$, contradicting our assumption that it was less than $\frac{1}{100}$. Hence, the series does not converge uniformly.

Exercise 4. Determine if

$$\sum_{n=1}^{\infty} \frac{\cos(k)}{k}$$

converges.

Solution 4. I believe this can also be done by a careful application of the integral comparison test, but I will solve it using summation by parts. We will first bound the "integral" terms, which will come from summing $\cos(k)$. Let $C_k = \sum_{j=1}^{k} \cos(j) = \frac{1}{2}\left(\sum_{j=1}^{k} e^{ik} + \sum_{j=1}^{k} e^{-ik}\right)$. By the geometric sum formula, $\left|\sum_{j=1}^{k} e^{ik}\right| \leq \frac{2}{|1-e^{i}|}$ and $\left|\sum_{j=1}^{k} e^{-ik}\right| \leq \frac{2}{|1-e^{-i}|}$, so $|C_k| = \frac{1}{2}\left|\sum_{j=1}^{k} e^{ik} + \sum_{j=1}^{k} e^{-ik}\right| \leq C$ for some absolute constant C. The "derivative" terms $\left|\frac{1}{k} - \frac{1}{k+1}\right| = \frac{1}{k(k+1)} \leq \frac{1}{k^2}$. We now see that that the product of the "integral" terms and the "derivative" terms is bounded above by $\frac{C}{k^2}$, a summable sequence. As long as the boundary terms converge (and unless the divergence test fails, one should expect the boundary terms to always converge), we should expect summation by parts to show convergence.

To do this carefully, recall that the summation by parts formula tells us that for $N \geq M$,

$$\left|\sum_{k=M}^{N} \frac{\cos(k)}{k}\right| \leq \frac{|C_N|}{N} + \frac{|C_M|}{M} + \sum_{k=M}^{N-1} \left|\frac{1}{k} - \frac{1}{k+1}\right| |C_k| \leq \frac{2}{M} + C \sum_{k=M}^{N-1} \frac{1}{k^2}.$$

As each term converges to 0 in M (for the final sum, this is a consequence of the fact that $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges), we must have that $\sum_{k=1}^{N} \frac{\cos(k)}{k}$ is a Caucy sequence in N, and hence converges as well.

Exercise 5. For $a, b \geq 0$, let

$$F(a,b) = \int_{-\infty}^{\infty} \frac{dx}{x^4 + (x-a)^4 + (x-b)^4}.$$

For what values of $p \in (0, \infty)$ is

$$\int_0^1 \int_0^1 F(a,b)^p \; da \; db < \infty?$$

Hint: try to prove that when $a \leq b$, $b^{-3}c \leq F(a,b) \leq b^{-3}C$ for positive constant $c < C$.

Solution 5. Explicitly computing $F(a,b)$ seems rather difficult. Instead, we will attempt to prove the approximate bound in the hint, with the assumption in the hint that $a \leq b$. When trying to bound $F(a,b)$, we need to somehow capture both the decay of $g(x) = \frac{1}{x^4+(x-a)^4+(x-b)^4}$ in the denominator at for large values of x and that $g(x)$ is bounded for small values of x. We can approximate $g(x)$ with one of the $(x-c)^{-4}$ terms to get decay when x is large, but we have to be careful in which one we choose to avoid issues when x is small. Or at least, my first couple attempts went nowhere.

We have assumed that $0 \leq a \leq b$, and without loss of generality, we may further assume that $0 < a < b$, since what is left in the final integral we want to bound is negligible. This allows us to choose our large value approximations $(x-c)^{-4}$ to avoid ever hitting a pole, while being useful over almost the entire domain. We will choose to approximate $g(x)$ with $\frac{1}{(x-a)^4}$ when $x > a+b$ or $x < a-b$ and with $\frac{1}{b^4}$ when $x \in (a-b, a+b)$. Let $h(x)$ the function given by our approximation.

To be rigorous about the approximations, it is easy to see that when $x > a+b$, $(x-a)^4 < x^4 + (x-a)^4 + (x-b)^4$, $(x-a)^4 \geq (x-a)^4$ and $(x-a)^4 \geq (x-b)^4$. We know that $x > a+b > 2a$, so $\frac{-a}{x} \geq \frac{-1}{2}$, and hence $(1-a/x)^4 > \frac{1}{16}$. Therefore, $(x-a)^4 > \frac{x^4}{16}$. Therefore, $(x-a)^4 > C_0(x^4 + (x-a)^4 + (x-b)^4)$ for some small enough constant C_0. We can conclude by taking the reciprocal of these inequalities that $h(x) \approx g(x)$ for $x > a+b$. For $x < a-b$, we proceed similarly to see that $h(x) \approx g(x)$. For $x \in (a-b, a+b)$, we write $x^4 + (x-a)^4 + (x-b)^4 = (y+a)^4 + y^4 + (y+a-b)^4$ for $y = x-a \in (-b, b)$. The latter polynomial can be expanded to a sum of D (a large absolute constant) monomials of degree 4, made up y, a, or b, and hence can be bounded above by Db^4 and, as one of those terms must necessarily be b^4, it can be bounded below by b^4 itself. It follows that $g(x) \approx h(x)$ in that range as well, and hence that $\frac{1}{C}\int_{-\infty}^{\infty} h(x)\ dx \leq \int_{-\infty}^{\infty} g(x)\ dx \leq \int_{-\infty}^{\infty} h(x)\ dx$.

For me, validifying the approximations was the most difficult part of the problem, and the rest was smooth sailing. We see that $\int_{a+b}^{\infty} \frac{1}{(x-a)^4}\ dx + \int_{-\infty}^{a-b} \frac{1}{(x-a)^4} = \frac{C_1}{b^3}$ for some absolute constant C_1. We also see that $\int_{a-b}^{a+b} \frac{1}{b^4}\ dx = \frac{2}{b^3}$. Then $\int_{-\infty}^{\infty} g(x)\ dx \in (C_2 b^{-3}, C_3 b^{-3})$ for absolute constants C_2, C_3. I keep saying "absolute constants" to emphasize that they do not depend on a or b, which will be important for what happens next.

We now use this approximation in the integral we actually want to bound. Since $F(a,b) = F(b,a)$, we know that $\int_{[0,1]^2} F(a,b)^p\ da\ db = 2\int_{a<b} F(a,b)^p\ da\ db$. By the approximation we just proved, we see that

$$C_2 \int_{a<b} b^{-3p}\ da\ db \leq \int_{a<b} F(a,b)^p\ da\ db \leq \int_{a<b} b^{-3p}\ da\ db.$$

Therefore, our desired values of p are precisely those where $\int_{a<b} b^{-3p}\ da\ db < \infty$. Finally, we write $\int_{a<b} b^{-3p}\ da\ db = \int_0^1 b^{1-3p}\ db$, and note by the p test that it converges if and only if $1-3p > -1$, or equivalently, $p < \frac{2}{3}$. This is our final answer.

Exercise 6. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Let $b_n \in \mathbb{R}$ be an increasing sequence with $\lim_{n\to\infty} b_n = \infty$. Show that

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^{n} b_k a_k = 0.$$

Solution 6. This would be much easier if $\sum_{n=1}^{\infty} a_n$ converged absolutely. If this was the case, we could reach the desired conclusion easily using dominated convergence. But we do not have absolutely convergence, so we have to workd a little harder.

The summation by parts formula tells us that $\sum_{k=1}^{n} b_k a_k = b_n A_n - \sum_{k=1}^{n-1}(b_{k+1} - b_k)A_k$, where $A_k = \sum_{j=1}^{k} a_j$. Let $L = \sum_{n=1}^{\infty} a_n$. Fix $\varepsilon > 0$ and choose N sufficiently large so that $|A_n - L| < \varepsilon/4$ for all $n \geq N$. Choose $M \geq N$ sufficiently large so that for $m \geq M$,

$\left|\sum_{k=1}^{N} \frac{b_{k+1}-b_k}{b_m} A_k\right| < \varepsilon/4$ and $\frac{b_{N+1}}{b_m} < \varepsilon/(4L)$. Then for $m \geq M$

$$\begin{aligned}
\left|\frac{1}{b_m}\left(\sum_{k=1}^{m} b_k a_k\right)\right| &= \left|A_m - \sum_{k=1}^{N} \frac{b_{k+1}-b_k}{b_m} A_k - \sum_{k=N+1}^{m} \frac{b_{k+1}-b_k}{b_m} A_k\right| \\
&\leq \left|\sum_{k=1}^{N} \frac{b_{k+1}-b_k}{b_m} A_k\right| + \frac{\varepsilon}{4}\left|\sum_{k=N+1}^{m} \frac{b_{k+1}-b_k}{b_m}\right| + \left|A_m - \sum_{k=N+1}^{m} \frac{b_{k+1}-b_k}{b_m} L\right| \\
&\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4}\left|\frac{b_{m+1}-b_{N+1}}{b_m}\right| + \left|L\left(1 - \frac{b_m - b_{N+1}}{b_m}\right)\right| + |A_m - L| \\
&\leq \frac{3\varepsilon}{4} + \frac{Lb_{N+1}}{b_m} \leq \varepsilon
\end{aligned}$$

Since ε was arbitrary, we arrive at the desired conclusion.