THE POINCARÉ INEQUALITY ON CONVEX DOMAINS

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1. Introduction

The $L^p$ norm measures the mass of a function, but tells us nothing about the derivatives of a function. Very irregular functions can converge in $L^p$ to very smooth functions. We could equip $C^k[0,1]$ with a norm

$$||f||_{k,p} = \left( \sum_{m=0}^{k} \int_{\Omega} |\partial^m f |^p \right)^{1/p}.$$  

With this definition, convergence in $||·||_{k,p}$ would imply $L^p$ convergence for all derivatives of order up to $k$. However, $C^k$ equipped with this norm is not complete. For example, the family of functions

$$\phi_\varepsilon(x) = \begin{cases} 
\varepsilon^{-1}e^{1/(1-|x/\varepsilon|^2)} & x \in [-\varepsilon, \varepsilon] \\
0 & \text{otherwise}
\end{cases}$$

is well known to be smooth for any $\varepsilon > 0$. $\lim_{\varepsilon \to 0} f_\varepsilon^{(m)}$ converges in $L^p$ for any $m$, but $\lim_{\varepsilon \to 0} f_\varepsilon$ is not even continuous. We can expand $C^k[0,1]$ to be complete with this norm by adding weakly differentiable functions. Weak derivatives agree with strong derivatives on differentiable functions and satisfy properties we expect of derivatives but allow differentiation on a much larger class of functions. Adding weakly differentiable functions gives a Banach space $W^{k,p}$ called the $k,p$ Sobolev space where convergence in the Sobolev space norm $||·||_{k,p}$ implies $L^p$ convergence of the first $k$ derivatives. We will give formal definitions of the weak derivative and Sobolev space in Section 2 and prove that the Sobolev space is a Banach space.

The $W^{1,q}$ norm bounds the $L^q$ norm of both a function and its first derivative, so a natural question is whether we can relate the $L^q$ norm of a function in $W^{1,q}$ to the $L^q$ norm of its derivative. A simple example of such a relation is the classical Poincaré inequality, stating that there exists $C$ depending $n$ and $q$ satisfying

$$||u||_q \leq C||\nabla u||_q \text{ for all } u \in W^{1,q}(\Omega) \text{ such that } \int_{\Omega} u = 0.$$ 

Finding optimal values for the $C$ depending on $n$, $q$, and $\Omega$ is an area of historical interest. A 1960 paper by Payne and Weinberger found that when $\Omega$ is a convex domain of diameter $d$ and $q = 2$, $C = \frac{d}{\pi}$ is the optimal constant in Poincaré inequality [1]. In 2004, a paper by Acosta and Durán proved that the constant $C = \frac{d}{2}$ is optimal for $q = 1$ [2]. Their proofs followed a similar structure: they decompose the convex domain into finitely many convex subsets narrow in all but one direction, they prove a one dimensional weighted version of the inequality, apply the weighted inequality in each of the thin convex sets to prove the Poincaré inequality in the subsets, and

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finally add over all of the subsets to prove desired inequality on the entire domain. In Section 3 we will give their proofs of the decomposition of the convex domain in Lemma 3.2 and of the weighted inequalities in Lemma 3.4 and Lemma 3.7. The papers of Payne and Weinberger and of Acosta and Durán used similar proofs for the final step of applying the inequality in the thin convex sets, but specific respectively to the $q = 2, q = 1$ case. We will generalize their proof to hold for any $q \in (1, \infty)$ in Theorem 3.3. A similar proof was found independently for the same result by Ferone, Nitsch, and Trombetti in 2012 [5]. We will use these results in Corollary 3.5 to prove that for convex domains $\Omega \subset \mathbb{R}^n$,

$$
\|u\|_2 \leq \frac{\text{diam}(\Omega)}{\pi} \|\nabla u\|_2 \quad \text{for all } u \in W^{1,2}(\Omega) \text{ such that } \int_{\Omega} u = 0 \quad (1.1)
$$

and in Corollary 3.8 to prove that

$$
\|u\|_1 \leq \frac{\text{diam}(\Omega)}{2} \|\nabla u\|_1 \quad \text{for all } u \in W^{1,1}(\Omega) \text{ such that } \int_{\Omega} u = 0. \quad (1.2)
$$

1.1. Notation. $\Omega$ will be a subset of $\mathbb{R}^n$, with additional conditions stated as necessary. We will generally use $q$ as the parameter for the $L^q$ and $W^{k,q}$ space to avoid confusion with convex functions denoted as $p$. We will denote the $L^q(\Omega)$ norm as $\|\cdot\|_q$. We will denote the $W^{k,q}(\Omega)$ norm as $\|\cdot\|_{k,q}$. For a $C^k(\Omega)$ function $f$ and a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, where $\alpha_i \geq 0$, we will denote by

$$
\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} = D^\alpha f.
$$

We will write $|\alpha| = \sum_{i=1}^n \alpha_i$. We will denote the multi-index $(0, 0, \ldots, 0)$ as 0. This index satisfies $D^0 u = u$. We also denote $\nabla u = [\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}]$. The space $C^k(\Omega)$ refers to the space of compactly supported $C^k(\Omega)$ functions. The space $L^1_{\text{loc}}(\Omega)$ is defined to be all $L^1$ functions supported on a compact subset of $\Omega$. For $q \in (1, \infty)$, $q^*$ will denote the Hölder conjugate of $q$: $\frac{1}{q} + \frac{1}{q^*} = 1$. Finally, $\alpha(n)$ will denote the $n$-dimensional Lebesgue measure of the unit ball.

2. Sobolev Spaces

To define Sobolev spaces, we first need to give a rigorous definition of the weak derivative. For $f \in C^1(\mathbb{R})$ and $g \in C^0_0(\mathbb{R})$, integration by parts gives that

$$
\int f g' \, dx = -\int f' g \, dx + \int (fg')' \, dx = -\int f' g \, dx + fg|_{-\infty}^{\infty} = -\int f' g \, dx.
$$

This generalizes for $C^k$ functions on $\mathbb{R}^n$: if $|\alpha| \leq k$, then

$$
\int f(D^\alpha g) \, dx = (-1)^{|\alpha|} \int (D^\alpha f)g \, dx.
$$

We require the weak derivative $D^\alpha f$ of an $L^1_{\text{loc}}(\Omega)$ function to satisfy this property for any smooth, compactly supported $g$.

**Definition 2.1.** The $\alpha$-weak derivative of a function $u \in L^1_{\text{loc}}(\Omega)$ is defined to be a function $v \in L^1_{\text{loc}}(\Omega)$ if

$$
\int_{\Omega} \varphi v \, dx = (-1)^{|\alpha|} \int_{\Omega} u D^\alpha \phi \, dx \text{ for all } \phi \in C^\infty_0(\Omega).
$$

With the weak derivative defined, we can define the Sobolev space:
Definition 2.2. For $q \geq 1$, $k$ a nonnegative integer, we define the $k,q$-Sobolev space

$$W^{k,q}(\Omega) = L^q(\Omega) \cap \{ u : ||u||_{k,q} < \infty \}$$

where

$$||u||_{k,q} = \left( \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^q \right)^{1/q}.$$

This norm is equivalent to $\sum_{|\alpha| \leq k} ||D^\alpha u||_q$. Since $(\int_\Omega |D^0 u|^q)^{1/q} = ||u||_q$, $||u||_{k,q} \geq ||u||_q$. We noted that $C^k(\Omega)$ is not complete in the introduction. We will now prove that $W^{k,q}$ is a Banach space.

Proposition 2.3. For any $\Omega \subset \mathbb{R}^n$, $W^{k,q}(\Omega)$ is a Banach space.

Proof. Clearly $W^{k,q}$ is a vector space, $||u||_{k,p} \geq 0$, and $||u||_{k,q} = 0$ if and only if $u = 0$. For $a \in \mathbb{R}$, $D^\alpha (au) = aD^\alpha u$, so $\int_\Omega |D^\alpha (au)|^p = |a|^p \int_\Omega |D^\alpha u|^q$ and hence $||(au)||_{k,q} = |a|||u||_{k,q}$. Finally, applying Minkowski’s inequality, we have

$$||u + v||_{k,q} = \left( \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha (u + v)|^p \right)^{1/q} = \left( \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u + D^\alpha v|^p \right)^{1/q} \leq \left( \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^q \right)^{1/q} + \left( \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha v|^q \right)^{1/q} \leq ||u||_{k,q} + ||v||_{k,q}.$$

And hence $|| \cdot ||_{k,q}$ satisfies the triangle inequality. It follows that $|| \cdot ||_{k,q}$ is a norm. It remains to prove that $W^{k,q}$ is complete with respect to $|| \cdot ||_{k,q}$. Let $f_n \in W^{k,q}$ be a Cauchy sequence. Since $f_n \in L^q$ and $||f||_q \leq ||f||_{k,q}$, $f_n$ is Cauchy in $L^q$ and hence has a limit $f \in L^q$. To complete the proof, we must show that for any multi-index $\alpha$ of order at most $k$, $f$ has weak derivatives $D^\alpha f$ and $D^\alpha f_n \to D^\alpha f$ in $L^q$.

Fix a multi-index $\alpha$ of order at most $k$. We know $D^\alpha f_n$ is Cauchy in $L^q$ since $f_n$ is Cauchy in $W^{k,q}$. Therefore there exists $g \in L^q$ such that $||D^\alpha f_n - g||_q \to 0$. We will prove that $g = D^\alpha f$. Fix $\phi \in C_0^\infty(\Omega)$. By the definition of the weak derivative, $\int_\Omega (D^\alpha \phi) f_n = (-1)^{|\alpha|} \int_\Omega \phi (D^\alpha f_n)$. By Hölder’s inequality and $D^\alpha \phi||_{p^*} < \infty$,

$$\int_\Omega |f_n - f| (D^\alpha \phi) \leq ||f_n - f||_q ||D^\alpha \phi||_{p^*} \to 0.$$

Similarly,

$$\int_\Omega |D^\alpha f_n - g| \phi \leq ||D^\alpha f_n - g||_q ||\phi||_{p^*} \to 0,$$

then

$$\lim_{n \to \infty} \int_\Omega f_n (D^\alpha \phi) = \int_\Omega f (D^\alpha \phi) \quad \text{and} \quad \lim_{n \to \infty} \int_\Omega (D^\alpha f_n) \phi = \int_\Omega g \phi.$$
And finally
\[ (-1)^{[\alpha]} \int_{\Omega} g \phi = \lim_{n \to \infty} (-1)^{[\alpha]} \int_{\Omega} (D^\alpha f_n) \phi = \lim_{n \to \infty} \int_{\Omega} (D^\alpha f_n) \phi = \int_{\Omega} (D^\alpha f) \phi. \]
Since this holds for any \( \phi \), \( D^\alpha f = g \) satisfies \( \|D^\alpha f_n - D^\alpha f\|_q \to 0 \), as desired. \( \square \)

3. POINCARE INEQUALITIES FOR BOUNDED CONVEX DOMAINS

There are many different versions of the Poincaré inequality. We will state a simple formulation here, see [3, pgs. 177-214] for other inequalities.

**Theorem 3.1.** Let \( \Omega \subset \mathbb{R}^n \) be bounded, connected, and open, such that \( \partial \Omega \) is \( C^1 \). Then there exists a constant \( C = C(n, p, \Omega) \) such that for \( u \in W^{1,p}(\Omega) \) with average value 0,
\[ \|u\|_p \leq C \|\nabla u\|_p. \]

A proof of this is given in Evans textbook *Partial Differential Equations* [4, 5.8.1]. Evans’ proof is nonconstructive, so other techniques are required to find values for the constant \( C \). We will restrict ourselves to convex, bounded domains. We will give conditions for an inequality of the form \( \|u\|_q \leq d/C \|\nabla u\|_q \) to be satisfied on a bounded, convex set \( \Omega \) in Theorem 3.3. We will then prove that the conditions are satisfied for the \( q = 2 \) case following Payne and Weinberger’s approach and for the \( q = 1 \) case following.

We first require Lemma 3.2, which proves that for a function \( u \) with average 0 on a domain \( \Omega \) and any \( \delta > 0 \), there is a decomposition of \( \Omega \) into convex sets \( \Omega_i \) such that \( u \) averages 0 on each \( \Omega_i \), and, for all but one direction, \( \Omega_i \) falls between parallel hyperplanes separated by \( \delta \). Our strategy will be to find a hyperplane which divides \( \Omega \) into two sets of equal measure, such that \( u \) averages 0 on both. We will repeat this process until the measure of each subset is small enough that it fits between parallel hyperplanes of distance \( \delta \) apart. We will do this again on each of the subsets, but restrict the orientation of the hyperplanes to be normal to the first hyperplanes we chose. We iterate this until we each convex set falls between \( n - 1 \) such pairs of parallel hyperplanes.

**Lemma 3.2.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded convex domain such that \( \int_{\Omega} u = 0 \). Then for any \( \delta > 0 \), there exists a finite disjoint collection of convex domains \( \{\Omega_i\} \) such that
\[ \overline{\Omega} = \bigcup \overline{\Omega}_i \quad \text{and} \quad \int_{\Omega_i} u = 0. \]
and for each \( \Omega_i \), there is an orthogonal coordinate system on \( \mathbb{R}^n \) such that
\[ \Omega_i \subset \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq \text{diam}(\Omega), 0 \leq x_j \leq \delta \text{ for all } j = 2, \ldots, n\}. \]

**Proof.** Consider the set of all vectors of the form \( v_\alpha = (0, \ldots, 0, \cos \alpha, \sin \alpha) \) and let \( P_\alpha \) be plane perpendicular to \( v_\alpha \) dividing \( \Omega \) into sets of equal area. Take some \( p \in P_\alpha \) and let
\[ \Omega_+ (\alpha) = \{x \in \Omega : (x - p) \cdot \alpha > 0\} \quad \Omega_- (\alpha) = \{x \in \Omega : (x - p) \cdot \alpha < 0\} \]
denote the two sets \( P_\alpha \) divides \( \Omega \) into. Let \( I(\alpha) = \int_{\Omega (\alpha)} u \). Since \( \int_{\Omega} u = 0 \), \( I(\alpha) + I(\alpha + \pi) = 0 \), so by continuity of \( I \), there exists some \( \alpha \) such that \( I(\alpha) = 0 \) and hence \( \int_{\Omega_+ (\alpha)} u = 0 \). We then repeat this process on \( \Omega_+, \Omega_- \) to get convex sets \( \Omega_i \) of arbitrarily small measure. Let \( r \) be the radius of the largest ball contained in \( \Omega_i \).
Then $\alpha(n)r^n \leq \mathcal{L}(\Omega_i)$, so by making the measure of $\Omega_i$ small, we can ensure $r < \frac{\delta}{2}$.

It is known that for convex sets with this property, there is a direction $\alpha$ and two parallel hyperplanes normal to some $\beta_i$ separated by distance $2r < \delta$ such that $\Omega_i$ is contained between them. For each $i$ we can then define a coordinate system with the last coordinate in the $\beta_i$ direction such that $\Omega_i \subset \mathbb{R}^{n-1} \times [0, \delta]$. We repeat this process on each $\Omega_i$ using the coordinate system defined in the previous step and looking at vectors of the form $v_u = (0, \ldots, 0, \cos \alpha, \sin \alpha, 0)$ until we arrive at a collection of convex sets $\Omega_{i,j}$ on which $u$ averages 0 of sufficiently small measure that for each $\Omega_{i,j}$, we can find a direction $\beta_{i,j}$ normal to $\beta_i$ such that $\Omega_{i,j}$ is contained between parallel hyperplanes normal to $\beta_{i,j}$ of distance $\leq \delta$. We then define a coordinate system such that $\beta_{i,j}, \beta_i$ are the last two coordinates. We repeat this process until we arrive at a finite collection of convex sets $\Omega_i$ with the desired properties.

We will use this in the main proof.

**Theorem 3.3.** Let $q \in [1, \infty)$, $\Omega \subset \mathbb{R}^n$ convex with diameter $d$. Suppose there exists $C$ such that for all $L \geq 0$, all $f \in W^{1,q}(0, L]$, and all non-negative $p : [0, L] \to \mathbb{R}$ such that $p^{1/m}$ is concave for some $m$ and $\int_0^L f(x)p(x) \, dx = 0$,

$$\int_0^L p|f'|^q \, dx \geq \left(\frac{C}{L}\right)^q \int_0^L p|f|^q \, dx. \quad (3.1)$$

Then for any $u \in W^{1,q}(\overline{\Omega})$ satisfying $\int_\Omega u = 0$,

$$||u||_q \leq (d/C)||\nabla u||_q.$$

**Proof.** If $\Omega$ has Lebesgue measure zero, this is clearly true. We will assume otherwise.

Since $C^\infty(\overline{\Omega})$ is dense in $W^{1,q}(\overline{\Omega})$, we can assume $u \in C^\infty(\overline{\Omega})$. By the compactness of the domain, there exists $M$ such that $|u| \leq M$ and $|(D^\alpha u)^q| \leq M$ for all $\alpha$ such that $|\alpha| \leq 2$ on $\Omega$. Fix $\delta > 0$. We first use Lemma 3.2 to divide $\Omega$ into convex subdomains $\Omega_i$, each with an orthogonal coordinate system $(x, y_1, \ldots, y_{n-1})$ such that $y_1, \ldots, y_{n-1} \in [0, \delta]$ and $x \in [0, d_i]$ for some $d_i \leq d$.

In each $\Omega_i$, define $p(x_0) = \mathcal{L}^{n-1}(\Omega_i \cap \{(x, y) : x = x_0\})$. Since measures are non-negative, $p$ is non-negative and, since $\Omega$ is not null, $p$ is positive on a set of positive measure. By the Brunn-Minkowski inequality, $p^{1/(n-1)}$ is concave. We will give estimates of $\int_{\Omega_i} |u|^q$ in terms of $\int_0^{d_i} p(x)|u(x, 0)|^q \, dx$, $\int_\Omega |\frac{\partial u}{\partial x}|^q$ in terms of $\int_0^{d_i} |\frac{\partial u}{\partial x}(x, 0)|^q \, dx$, and $\int_\Omega u$ in terms of $\int_0^{d_i} p(x)u(x, 0)^q \, dx$, at which point we can apply the inequality (3.1) for fixed $(x, y_1, \ldots, y_{n-1})$, applying the mean value theorem to $|u|^q$ in the $y_1$ direction gives that

$$||u(x, 0, y_2, \ldots, y_n) - u(x, y_1, y_2, \ldots, y_{n-1})||_q < M\delta.$$  

We can do the same for $y_2, y_3, \ldots, y_{n-1}$ directions and sum to get

$$||u(x, y)^q - |u(x, 0)|^q||_q \leq (n - 1)M\delta.$$  

Integrating this inequality over $\Omega_i$ gives

$$\int_0^{d_i} \int_{\Omega_i \cap \{\{x\} \times \mathbb{R}^{n-1}\}} |u(x, y)|^q \, dx \, dy - \int_0^{d_i} p(x)|u(x, 0)|^q \, dx \leq (n - 1)M|\Omega_i|\delta.$$
By Fubini’s theorem, $\int_{\Omega} |u|^q = \int_{\Omega} \int_{\{x \times \mathbb{R}^{n-1}\}} |u(x, y)|^q \, dy \, dx$, so

$$\left| \int_{\Omega_i} |u|^q - \int_0^{d_i} p(x) |u(x, 0)|^q \, dx \right| \leq (n - 1) M |\Omega_i| \delta$$  \hspace{1cm} (3.2)

Applying the same technique to $|\partial u / \partial x|$ and $u$ gives

$$\left| \int_{\Omega_i} \left| \frac{\partial u}{\partial x}(x, y) \right|^q \, dx \, dy - \int_0^{d_i} p(x) \left| \frac{\partial u}{\partial x}(x, 0) \right|^q \, dx \right| < (n - 1) M |\Omega_i| \delta,$$  \hspace{1cm} (3.3)

and

$$\left| \int_{\Omega} u(x, y) \, dx \, dy - \int_0^{d_i} p(x) u(x, 0) \, dx \right| \leq (n - 1) M |\Omega_i| \delta.$$  \hspace{1cm} (3.4)

By (3.4) and $\int_{\Omega_i} u \, dx = 0$, we have that

$$\left| \int_0^{d_i} u(x, 0) p(x) \, dx \right| \leq (n - 1) M |\Omega_i| \delta.$$

Set

$$f(x) = u(x, 0) - \frac{\int_0^{d_i} u(y, 0) p(y) \, dy}{\int_0^{d_i} p(y) \, dy}$$

By construction $\int_0^{d_i} f(x) p(x) \, dx = 0$ and hence $\int_0^{d_i} |f|^q p \, dx \geq (C/d)^q \int_0^{d_i} |f|^q p \, dx$. Then

$$\int_0^{d_i} |u(x, 0)|^q p \, dx \leq (d_i/C)^q \int_0^{d_i} \left| \frac{\partial u}{\partial x}(x, 0) \right|^q p(x) \, dx + \int_0^{d_i} \left| \frac{\int_0^{d_i} u(y, 0) p(y) \, dy}{\int_0^{d_i} p(y) \, dy} \right|^q p(x) \, dx.$$  \hspace{1cm} (3.5)

Since $\left| \int_0^{d_i} u(x, 0) p(x) \, dx \right| < (n - 1) M |\Omega_i| \delta$,

$$\left| \int_0^{d_i} u(y, 0) p(y) \, dy \right|^q \leq [(n - 1) M |\Omega_i| \delta]^q$$

and thus, letting $P = \left( \int_0^{d_i} p(x) \, dx \right)^{1-q}$ < $\infty$,

$$\int_0^{d_i} \left| \frac{\int_0^{d_i} u(y, 0) p(y) \, dy}{\int_0^{d_i} p(y) \, dy} \right|^q p(x) \, dx \leq P [(n - 1) M |\Omega_i| \delta]^q.$$

Combining this with (3.5), we have

$$\int_0^{d_i} |u(x, 0)|^q p \, dx \leq (d_i/C)^q \int_0^{d_i} \left| \frac{\partial u}{\partial x}(x, 0) \right|^q p(x) \, dx + P [(n - 1) M |\Omega_i| \delta]^q,$$  \hspace{1cm} (3.6)

Applying (3.3) and $|\partial u / \partial x|^q \leq |\nabla u(x, y)|^q$, we see that

$$\int_0^{d_i} |u(x, 0)|^q p \, dx \leq (d_i/C)^q \int_0^{d_i} \left| \frac{\partial u}{\partial x}(x, y) \right|^q p(x) \, dx + ((d_i/C)^q + P) [(n - 1) M |\Omega_i| \delta]^q,$$

$$\leq (d_i/C)^q \int_0^{d_i} |\nabla u(x, y)|^q p(x) \, dx + ((d_i/C)^q + P) [(n - 1) M |\Omega_i| \delta]^q.$$

$$\leq (d_i/C)^q \int_0^{d_i} |\nabla u(x, y)|^q p(x) \, dx + ((d_i/C)^q + P) [(n - 1) M |\Omega_i| \delta]^q.$$
Applying (3.2), we have
\[ \int_{\Omega_i} |u|^q \, dx \, dy \leq (d_i/C)^q \int_{\Omega_i} |\nabla u(x, y)| \, dx \, dy + ((d_i/C)^q + 1 + P) [(n - 1) M |\Omega_i| \delta] \]
Since \( d_i \leq d \), \( \int_{\Omega_i} |u(x, y)|^q \, dx \, dy \leq (d/C)^q \int_{\Omega_i} |\nabla u(x, y)|^q \, dx \, dy + ((d/C)^q + 1 + P) [(n - 1) M |\Omega_i| \delta] \)

Summing over all \( i \) gives
\[ \int_{\Omega} |u(x, y)|^q \, dx \, dy \leq (d/C)^q \int_{\Omega} |\nabla u(x, y)|^q \, dx \, dy + ((d/C)^q + 1 + P) [(n - 1) M |\Omega| \delta] \]

Taking \( \delta \) to 0 gives
\[ \int_{\Omega} |u|^q \leq (d/C)^q \int_{\Omega} |\nabla u|^q \]
and hence
\[ ||u||_q \leq (d/C)||\nabla u||_q. \]

\[ \square \]

To complete the proofs of the inequalities (1.1) and (1.2), it suffices to prove that the inequality (3.1) holds under the assumptions of Theorem 3.3. We will, following chronological order, start with the \( L^2 \) estimate.

This proof relies heavily on the theory of Sturm-Liouville systems. We will first introduce that area, then move to the \( L^2 \) estimate proof. We need that a function \( f \) minimizing \( \int_0^L \frac{p(v)^2}{\int_0^L p v^2} \) over the set of functions satisfying \( \int_0^L p(y)v(y) \, dy = 0 \) solves the Sturm-Liouville system
\[
\begin{align*}
[pf']' + \lambda pf &= 0 \\
f'(0) &= f'(L) = 0,
\end{align*}
\]
where \( \lambda \) is the minimum value of \( \int_0^L \frac{p(v)^2}{\int_0^L p v^2} \).

Minimizing \( \int_0^b \frac{p(f')^2}{\int_a^b pf'^2} \) for some \( p \in C^1[a, b] \) over \( f \) satisfying \( \int_a^b fp = 0 \) is equivalent to minimizing \( \int_0^b p(f')^2 \) with the constraint \( \int_a^b pf'^2 = 1 \) over the set of \( f \in W^{1,2}[a, b] \) subject to \( \int_a^b p(y)f(y) \, dy = 0 \). Then using the Euler-Lagrange equation, a minimum \( f \) must satisfy \( [pf']'' + \lambda pf = 0 \). Set
\[
v_a(x) = \begin{cases} \frac{1}{p(a)} & x = a \\ 0 & x \neq a \end{cases}, \quad v_b(x) = \begin{cases} -\frac{1}{p(b)} & x = b \\ 0 & x \neq b \end{cases}.
\]

Then \( \int_a^b [pf']'v_1 = \int_a^b \alpha pf v_1 \). Approximating \( v_1, v_0 \) by differentiable functions and integratogens by parts gives \( pf'v_1|_a^b - \int_a^b pf'v_1 = \alpha \int_a^b pf v_1 \). The integrals and \( (pf'v_1)(0) \) all vanish, what remains is \( f'(L) = 0 \). Similar reasoning for \( v_0 \) gives that \( f'(0) = 0 \).

Sturm-Liouville systems of the form (3.7) are known to satisfy \( -\alpha = \inf \int_0^L \frac{p(v)^2}{\int_0^L p v^2} \),

where the infimum is again over functions satisfying \( \int_0^L p(y)v(y) \, dy = 0 \). It follows that \( -\lambda = \alpha \) and hence the minimizer \( f \) satisfies \( [pf']'' + \lambda pf = 0 \) subject to the boundary conditions \( f'(a) = f'(b) = 0 \).

The proof given below is slightly modified from the proof given in Payne, which erroneously required \( p \) itself to be concave. J. P. Pinasco noted that for dimension \( n > 2 \), \( p \) as defined in 3.3 is not concave and communicated that to the authors of
Acosta and Durán, who made the necessary adjustments to their own proof. I will note the modification from Payne’s proof.

**Lemma 3.4.** Let \( p : [0, L] \) be a non-negative function such that \( p^{1/m} \) is concave for some \( m \). Then for any \( f \in W^{1,2}([0, L]) \) satisfying \( \int_0^L p(x)f(x) \, dx = 0 \),

\[
\int_0^L p(x)[f'(x)]^2 \, dx \geq \frac{\pi^2}{L^2} \int_0^L p(x)f(x)^2 \, dx.
\]

**Proof.** We will first assume \( p \) is twice differentiable and strictly positive.

If \( \lambda \) minimizes \( \frac{\int_0^L p(f')^2}{\int_0^L p f^2} \), then the minimizing function \( f \) satisfies

\[
[pf']' + \lambda pf = 0 \tag{3.8}
\]

\[
f'(0) = f'(L) = 0
\]

We can divide by \( p \) and differentiate again to get

\[
\left[ pf'' \right]' + pf' + \lambda f = 0.
\]

Making the substitution \( w = f'/\sqrt{p} \) in 3.7, we arrive at a new Sturm-Liouville system:

\[
w'' + \alpha w + \lambda w = 0 \tag{3.9}
\]

\[
w(0) = w(L) = 0,
\]

where \( \alpha = \left[ \frac{1}{2} \frac{p''}{p} - \frac{3}{4} \frac{(p')^2}{p^2} \right] \). What follows is slightly different from the proof in [1], which only required that \( p \) be concave. Since \( p^{1/m} \) is concave, \( \left[ p^{1/m}(v) \right]'' < 0 \) and hence \( \frac{p''}{p} - \frac{m-1}{2m} \frac{(p')^2}{p^2} < 0 \). This and \( \frac{m-1}{2m} \leq \frac{3}{4} \) implies that \( \frac{p''}{p} - \frac{3}{4} \frac{(p')^2}{p^2} < 0 \) and hence \( \alpha < 0 \). Then \( w'' + \lambda w > 0 \). Since \( w \geq 0 \), \( w''w + \lambda w^2 \geq 0 \), so

\[
\int_0^L w''w \, dy + \lambda \int_0^L w^2 \, dy \geq 0. \tag{3.10}
\]

Integrating by parts, we have

\[
\int_0^L w''w \, dy = (w')|_0^L - \int_0^L (w')^2 \, dy = - \int_0^L (w')^2 \, dy.
\]

Substituting this into 3.10 we see that

\[
\lambda \geq \frac{\int_0^L (w')^2 \, dy}{\int_0^L w^2 \, dy}. \tag{3.11}
\]

Consider the function \( g(w) = -w'' \) over smooth functions on \([0, L]\) which vanish at the endpoints. The Rayleigh quotient for \( g \) after integrating by parts is \( \inf \frac{\int_0^L (w')^2 \, dy}{\int_0^L w^2 \, dy} \) which equals the smallest eigenvalue of \( L, \pi^2 L^{-2} \). Hence

\[
\lambda \geq \pi^2 L^{-2}. \tag{3.12}
\]

We can uniformly approximate convex functions with smooth convex functions, so we have for any \( p \) such that \( p^{1/m} \) is convex, \( \int_0^L p(u')^2 \, dy \geq \pi^2 L^{-2} \int_0^L p u^2 \, dy \), as desired.

Using this result the desired constant in the \( q = 2 \) is immediate:
Corollary 3.5. For any $u \in W^{1,2}(\Omega)$ such that $\int_{\Omega} u = 0$,

$$||u||_2 \leq \frac{d}{\pi} ||\nabla u||_2.$$  

Proof. By Lemma 3.4 for any $u \in W^{1,2}$ such that for any non-negative $p$ such that $p^{1/m}$ convex for some $m$ and $\int_0 L p(y) u(y) \, dy = 0$, $\int_0 L p(y)[u'(y)]^2 \, dy \geq \frac{\pi^2}{L^2} \int_0 L p(y)[u(y)^2] \, dy$. Then by Theorem 3.3

$$||u||_2 \leq \frac{d}{\pi} ||\nabla u||_2.$$  

We know proceed with the $L^1$ estimate, following the proof given by Acosta.

Lemma 3.6. Let $p$ be a non-negative function on $[0, 1]$ such that $\int_0^1 p(x) \, dx = 1$ and such that, for some $m \in \mathbb{N}$, $p^{1/m}$ is concave. Then,

$$\frac{\int_0^x p(t) \, dt \int_0^1 p(t) \, dt}{p(x)} \leq \frac{1}{4}. \tag{3.13}$$

The proof for this lemma is fairly technical, refer to [2, Lemma 3.1] for its proof.

Lemma 3.7. Let $p : [0, L]$ be a non-negative function such that $p^{1/m}$ is concave for some $m$. Then for any $f \in W^{1,1}([0, L])$ satisfying $\int_0^L p(y)f(y) \, dy = 0$,

$$\int_0^L p(y)|f'(y)| \, dy \geq 2 \int_0^L p(y)|f(y)| \, dy.$$

Proof. We can rescale to assume $L = 1$ and $\int_0^1 p(x) \, dx = 1$. Integrating $f(x)p(x)$ by parts gives that

$$f(x)p(x) = f(1) \int_0^1 p(t) \, dt - f(0) \int_0^1 p(t) \, dt - \int_0^1 f'(x) \left( \int_0^x p(t) \, dt \right) \, dx.$$

Since $\int_0^1 p = 0$,

$$- \int_0^1 f'(x) \left( \int_0^x p(t) \, dt \right) \, dx$$

$$= - \int_0^y f'(x) \left( \int_0^x p(t) \, dt \right) \, dx - \int_y^1 f'(x) + \int_y^1 f'(x) \left( \int_1^x p(t) \, dt \right) \, dx$$

$$= f(y) - \int_0^y f'(x) \left( \int_0^x p(t) \, dt \right) \, dx + \int_y^1 f'(x) \left( \int_x^1 p(t) \, dt \right) \, dx.$$

Since $\int_0^1 f(x)p(x) \, dx = 0$, we have that

$$f(y) = \int_0^y f'(x) \left( \int_0^x p(t) \, dt \right) \, dx - \int_y^1 f'(x) \left( \int_1^x p(t) \, dt \right) \, dx.$$

It follows that

$$|f(y)| \leq \int_0^y |f'(x)| \left( \int_0^x p(t) \, dt \right) \, dx + \int_y^1 |f'(x)| \left( \int_x^1 p(t) \, dt \right) \, dx.$$  

Multiplying by $p(y)$ and integrating gives
\[ \int_0^1 |f(y)|p(y) \, dy \leq \int_0^1 \left[ \int_0^y |f'(x)| \left( \int_0^x p(t) \, dt \right) \, dx \right. + \int_y^1 |f'(x)| \left. \left( \int_x^1 p(t) \, dt \right) \, dx \right] \, dy. \]

Swapping the order of integration and simplifying yields

\[ \int_0^1 |f(y)|p(y) \, dy \leq 2 \int_0^1 |f'(x)| \left( \int_0^x p(t) \, dt \right) \left( \int_x^1 p(t) \, dt \right) \, dx. \]

We will prove that \( \frac{(\int_0^x p(t) \, dt)(\int_0^1 p(t) \, dt)}{p(x)} \) \( \leq \frac{1}{4} \) for all \( x \in (0,1) \) in Lemma 3.6, which will give that \( \int_0^1 |f(y)|p(y) \, dy \leq 2 \int_0^1 |f'(y)|p(y) \, dy \). Then applying the substitution \( y = x/L \) gives \( \frac{1}{L} \int_0^L |f(y)|p(y) \, dy \leq \frac{2}{L} \int_0^L |f'(y)|p(y) \, dy \) and thus, as desired,

\[ \int_0^L |f(y)|p(y) \, dy \leq \frac{2}{L} \int_0^L |f'(y)|p(y) \, dy. \]

\( \square \)

Again, the final constant follows easily from 3.3 and 3.7

**Corollary 3.8.** For any \( u \in W^{1,1}(\Omega) \) such that \( \int_{\Omega} u = 0 \),

\[ ||u||_1 \leq \frac{d}{2} ||\nabla u||_1. \]

**Proof.** By Lemma 3.4 for any \( u \in W^{1,1} \) such that for any non-negative \( p \) with \( p^{1/m} \) convex and \( \int_0^L p(y) u(y) \, dy = 0, \int_0^L p(y) |u(y)| \, dy \geq \frac{\pi}{L} \int_0^L p(y) |u(y)| \, dy \). Then by Theorem 3.3

\[ ||u||_1 \leq \frac{d}{2} ||\nabla u||_1. \]

\( \square \)

Both the \( L^1 \) and \( L^2 \) inequalities are sharp, as proven in [2] and [1] respectively. While this paper only considers the \( q = 1,2 \) case, sharp bounds have been found for all \( q \in [1,\infty) \):

\[ ||u||_q \leq \frac{d}{\pi q} ||\nabla u||_q, \quad \text{where} \quad \pi_q = 2\pi \frac{(q - 1)^{1/q}}{q \sin(\pi/q)}. \]

This was proven in [5].

4. Acknowledgements

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References


