# COMPLEX ANALYSIS PROBLEMS

**Exercise 1.** Let $n \in \mathbb{N}, n \geq 3$. Compute the integral
$$\int_0^\infty \frac{dx}{1+x^n}.$$

**Exercise 2.** Consider
$$f(z) = \sum_{n=0}^{\infty} z^{2n}$$
which defines an analytic function inside the unit disk $\mathbf{D}$.

(1) Let $w \in \mathbb{C}$ be such that $w^{2^\ell} = 1$ for some integer $\ell$. Does $\lim_{r\to 1} f(rw)$ exist?
(2) Can $f$ be extended analyticaly to some open domain $\Omega \supsetneq \mathbf{D}$?

**Exercise 3.** Consider
$$f_n(z) = 1 + \frac{1}{z} + \frac{1}{z^2 2!} + \cdots + \frac{1}{z^n n!}.$$
Prove that for every $\delta > 0$, there is $n \in \mathbb{N}$ such that all zeros of $f_n(z)$ are inside $\{|z| < \delta\}$.

**Exercise 4.** Let $f$ be analytic in the upper half plane and continuous on it's closure. Assume that $f$ satisfies the estimate $|f(z)| \leq M|z|^{-r}$, $z \neq 0$ for strictly positive constants $M$ and $r$. Show that if $\mathrm{Im}(z) > 0$, then
$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-z}\, dt.$$

**Exercise 5.** Recall that an entire function $f$ is said to be of exponential type if
$$|f(z)| \leq Ce^{D|z|}$$
for some positive constants $C$ and $D$. Prove that if $f$ is an entire function of exponential type, then so is $f'$.

**Exercise 6.** Let $f_n$ be a sequence of analytic functions in a complex domain $\Omega$. Suppose that all of $f_n$ are injective in $\Omega$ and that $f_n \to f$ uniformly on compact subsets of $\Omega$. Show that then eitehr $f$ is one-to-one in $\Omega$ or $f$ is constant.

**Exercise 7.** Show that any two bounded analytic functions in the strip $\{|\mathrm{Im} z| < 1/2\}$ which coincide on the set $\{1/\pi \log n : n \in \mathbb{N}\}$ must coincide on the whole strip. Can the same be said about the set $\{2/\pi \log n : n \in \mathbb{N}\}$?

**Exercise 8.** Suppose the functions $f_1$ and $f_2$ are analytic at each point of $\overline{\mathbf{D}}$. Prove that the function $|f_1(z)| + |f_2(z)|$, when considered on $\overline{\mathbf{D}}$, reaches its maximum on $\partial\mathbf{D}$

**Exercise 9.** Compute the improper integral
$$\int_{-\infty}^{\infty} \frac{e^{its^5} s^4}{1+s^{10}}\, ds$$
for every $t \in \mathbb{R}$.

**Exercise 10.** Suppose $f : \mathbf{D} \to \mathbb{C}$ is holomorphic. Show that there exists a sequence $z_n \in \mathbf{D}$ such that $\lim_{n\to\infty} |z_n| = 1$ and $\limsup_{n\to\infty} |f(z_n)| < \infty$.

**Exercise 11.** Fix $0 < r < R$. Prove that there exists $\varepsilon > 0$ such that if $P : \mathbb{C} \to \mathbb{C}$ is a polynomial, then

$$\max_{r \le |z| \le R} \left| P(z) - \frac{1}{z} \right| > \varepsilon.$$

*Note that $\varepsilon$ is independent of $P$.*

**Exercise 12.** Suppose $D \subset \mathbb{C}$ is a bounded domain and $f : D \to D$ is holomorphic. Prove that if $0 \in D$, $f(0) = 0$, and $f'(0) = 1$, then $f(z) = z$ for all $z \in D$.