Math 121A: Homework 9 solutions

1. The first integral is given by

$$\begin{aligned}
\oint_C z^2\,dz &= \int_{-1}^{1} x^2\,dx + \int_0^1 (1+iy)^2 i dy + \int_1^{-1} (i+x)^2 dx + \int_1^0 (-1+iy)^2 i dy \\
&= \left[\frac{x^3}{3}\right]_{-1}^{1} + \left[\frac{(1+iy)^3}{3}\right]_0^1 - \left[\frac{(i+x)^3}{3}\right]_{-1}^{1} - \left[\frac{(-1+iy)^3}{3}\right]_0^1 \\
&= \left(\frac{1}{3} - \frac{-1}{3}\right) + \left(\frac{(1+i)^3}{3} - \frac{1}{3}\right) \\
&\quad - \left(\frac{(i+1)^3}{3} - \frac{(i-1)^3}{3}\right) - \left(\frac{(-1+i)^3}{3} - \frac{(-1)^3}{3}\right) \\
&= 0,
\end{aligned}$$

since all terms cancel in pairs. The second integral is given by

$$\begin{aligned}
\oint_C z^2\,dz &= \int_{-1}^{1} x^2\,dx + \int_0^{\pi} (e^{i\theta})^2 i e^{i\theta} d\theta \\
&= \int_{-1}^{1} x^2\,dx + i\int_0^{\pi} e^{3i\theta} d\theta \\
&= \left[\frac{x^3}{3}\right]_{-1}^{1} + \left[\frac{e^{i\theta}}{3}\right]_0^{\pi} \\
&= \left(\frac{1}{3} - \frac{-1}{3}\right) + \left(\frac{-1}{3} - \frac{1}{3}\right) \\
&= 0.
\end{aligned}$$

As expected, both integrals evaluate to zero, since they involve integrating an analytic function around a closed curve.

2. Differentiating Cauchy's formula gives

$$f'(z) = \frac{df}{dz} = \frac{1}{2\pi i}\oint_C \frac{\partial}{\partial z}\left(\frac{f(w)}{w-z}\right) dw = \frac{1}{2\pi i}\oint_C \frac{f(w)\,dw}{(w-z)^2}.$$

Differentiating n times gives

$$\begin{aligned}
f^{(n)}(z) &= \frac{d^n f}{dz^n} \\
&= \frac{1}{2\pi i}\oint_C \frac{\partial^n}{\partial z^n}\left(\frac{f(w)}{w-z}\right) dw \\
&= \frac{1}{2\pi i}\oint_C \frac{\partial^{n-1}}{\partial z^{n-1}}\left(\frac{f(w)}{(w-z)^2}\right) dw \\
&= \frac{1}{2\pi i}\oint_c \frac{n!f(w)\,dw}{(w-z)^{n+1}}.
\end{aligned}$$

3. By making use of the result from the previous question, the integral can be evaluated according to

$$\begin{aligned}\oint_C \frac{\sin 2z\, dz}{(6z-\pi)^3} &= \frac{1}{216}\oint_C \frac{\sin 2z\, dz}{(z-\frac{\pi}{6})^3} \\ &= \frac{2\pi i}{2\times 216}\frac{d^2}{dz^2}\sin 2z\bigg|_{z=\pi/6} \\ &= \frac{\pi i}{216}\left(-4\sin\left(\frac{2\pi}{6}\right)\right) \\ &= -\frac{4\pi i}{216}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\pi i\sqrt{3}}{108}.\end{aligned}$$

4. By making the substitution $z = z_0 + \rho e^{i\theta}$, the integral can be written as

$$\oint_C \frac{dz}{(z-z_0)^n} = \int_0^{2\pi} \frac{ie^{i\theta}d\theta}{e^{ni\theta}} = i\int_0^{2\pi} e^{(n-1)^\theta} d\theta.$$

If $n = 1$ the integrand is 1, and thus the integral evaluates to $2\pi i$. Otherwise the integral is

$$i\int_0^{2\pi} e^{(n-1)^\theta} d\theta = i\left[\frac{e^{(n-1)\theta}}{n-1}\right]_0^{2\pi} = \frac{i}{n-1} - \frac{i}{n-1} = 0.$$

5. By writing $y = z - 1$, the Laurent series at $z = 1$ is given by

$$\frac{e^z}{z^2-1} = \frac{e^{y+1}}{2y\left(1+\frac{y}{2}\right)} = \frac{e}{y}\left(\sum_{m=0}^{\infty}\frac{y^m}{m!}\right)\left(\sum_{n=0}^{\infty}\frac{y^n}{(-2)^n}\right).$$

While it is difficult to write down an explicit expression for terms in the Laurent series, the first three terms are given by

$$\begin{aligned}\frac{e^z}{z^2-1} &= \frac{e}{y}\left(1+y+\frac{y^2}{2}+\dots\right)\left(1-\frac{y}{2}+\frac{y^2}{4}+\dots\right) \\ &= \frac{e}{y}\left(1+\frac{y}{2}+\frac{y^2}{4}+\dots\right).\end{aligned}$$

While these first three terms in the bracket appear to agree with the pattern $\sum_{n=0}^{\infty} y^n 2^{-n}$, this is coincidental, and later terms in the series do not follow this pattern.

6. By writing $y = z - 2$, the Laurent series at $z = 2$ is given by

$$\frac{1}{z^2-5z+6} = \frac{1}{(z-2)(z-3)} = -\frac{1}{y(1-y)} = -\frac{1}{y}\sum_{n=0}^{\infty} y^n = -\sum_{k=-1}^{\infty}(z-2)^k.$$

7. By making the substitution $e^{i\theta} = z$, the integral be rewritten as an integral around the unit circle C,

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{13 + 5\sin\theta} &= \int_0^{2\pi} \frac{1}{13 + \frac{5}{2i}(e^{i\theta} - e^{-i\theta})} \\
&= \oint_C \left(\frac{dz}{iz}\right)\left(\frac{2}{26 - 5iz + 5iz^{-1}}\right) \\
&= \oint_C \frac{2dz}{5z^2 + 26iz - 5} \\
&= \oint_C \frac{2dz}{(5z+i)(z+5i)}.
\end{aligned}$$

The integrand has simple poles at $z = -5i$ (which is outside C) and $z = -i/5$ (which is inside C). By the residue theorem, the integral can be evaluated in terms of the residue at $z = -i/5$,

$$\begin{aligned}
\text{Res}\left(\frac{2}{(5z+i)(z+5i)}, z = -\frac{i}{5}\right) &= \lim_{z \to -i/5} \frac{2(z + i/5)}{(5z+i)(z+5i)} \\
&= \lim_{z \to -i/5} \frac{2}{5(z+5i)} \\
&= \frac{2}{5i(-1/5 + 5)} = \frac{1}{12i}.
\end{aligned}$$

Hence the integral is given by

$$\int_0^{2\pi} \frac{d\theta}{13 + 5\sin\theta} = 2\pi i \left(\frac{1}{12i}\right) = \frac{\pi}{6}.$$