Math 121A: Homework 8 solutions

1. (a) To calculate a Green function solution, consider the case of an impulsive input, $f(x) = \delta(x - x')$ for $-1 < x' < 1$. Then for $x < x'$ and $x > x'$ the solution has the form

$$y(x) = Ax + B$$

for some constants A and B, which are different in the two regions. In the region $x < x'$, the solution must satisfy the boundary condition $y(-1) = 0$, and hence

$$y_-(x) = C(x + 1)$$

for some constant C. Similarly, in the region $x > x'$, in order to satisfy the boundary condition $y(1) = 0$, the solution is

$$y_+(x) = D(x - 1)$$

for some constant D. To satisfy continuity at $x = x'$,

$$C(x' + 1) = D(x' - 1)$$

and to ensure a change of derivative of 1 at $x = x'$,

$$1 = y_+'(x') - y_-'(x') = D - C.$$

Hence

$$C(x' + 1) = (C + 1)(x' - 1)$$

and thus

$$C = \frac{x' - 1}{2}, \qquad D = \frac{x' + 1}{2}.$$

The Green function is therefore

$$G(x, x') = \begin{cases} \frac{(x+1)(x'-1)}{2} & \text{for } x < x', \\ \frac{(x-1)(x'+1)}{2} & \text{for } x > x'. \end{cases}$$

(b) The functions are shown in Fig. 1. For each value of x' the solution corresponds the case when $f(x)$ is given by an impulsive term, $\delta(x - x')$. Each solution is continuous, and the delta function term creates a shift in derivative of size 1 at x'.

Physically, the equation that is being solved can be thought of as describing the position of an elastic string that is held fixed at $x = \pm 1$, where forces are applied along its length. The curves in Fig. 1 are what would be expected if a point mass was attached to a string and it was allowed to deform.

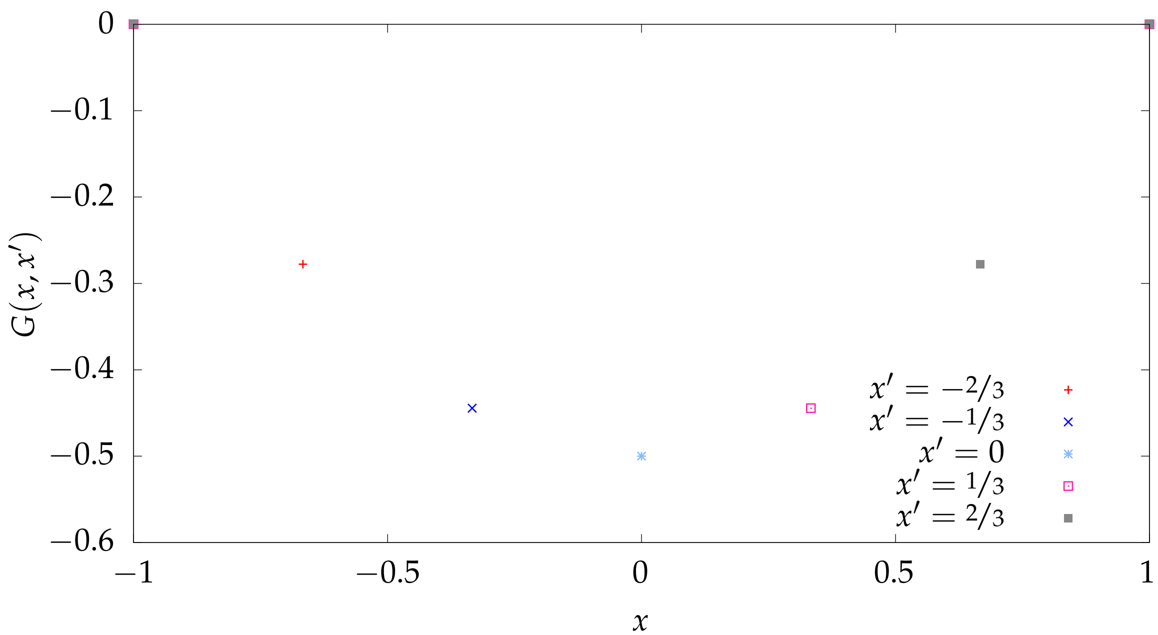

Figure 1: Several solutions $G(x, x')$ representing solutions of the differential equation for the case when $f(x) = \delta(x - x')$.

(c) The solution is given by

$$y(x) = \int_{-1}^{1} G(x,x')f(x')dx' = \int_{-1/4}^{1/4} G(x,x')dx'$$

and there are several cases depending on the value of x. If $x < -1/4$,

$$\begin{aligned} y(x) &= \int_{-1/4}^{1/4} \frac{(x+1)(x'-1)}{2} dx' \\ &= \frac{(x+1)}{2} \int_{-1/4}^{1/4} (x'-1)dx' \\ &= \frac{(x+1)}{2} \left[\frac{x'^2}{2} - x' \right]_{-1/4}^{1/4} \\ &= -\frac{x+1}{4}. \end{aligned}$$

By symmmetry, the solution for $x > 1/4$ is given by

$$y(x) = \frac{x-1}{4}.$$

If $|x| \leq 1/4$, then

$$\begin{aligned} y(x) &= \int_{-1/4}^{x} G(x,x')f(x')dx' + \int_{x}^{1/4} G(x,x')f(x')dx' \\ &= \frac{x-1}{2} \int_{-1/4}^{x} (x'+1)dx' + \frac{x+1}{2} \int_{x}^{1/4} (x'-1)dx' \\ &= \frac{x-1}{2} \left[\frac{x'^2}{2} + x' \right]_{-1/4}^{x} + \frac{x+1}{2} \left[\frac{x'^2}{2} - x' \right]_{x}^{1/4} \\ &= \frac{x^2}{2} + \frac{7}{32}. \end{aligned}$$

Hence the general solution can be written as

$$y(x) = \begin{cases} \frac{x^2}{2} + \frac{7}{32} & \text{for } |x| < 1/4, \\ \frac{|x|-1}{4} & \text{for } |x| \geq 1/4. \end{cases}$$

The input function $f(x)$ and solution $y(x)$ are shown in Figs. 2(a) and 2(b) respectively. It can be seen that the solution has a similar form to the solutions shown in Fig. 1, and is a combination of the Green functions $G(x,x')$ for $|x'| < 1/4$.

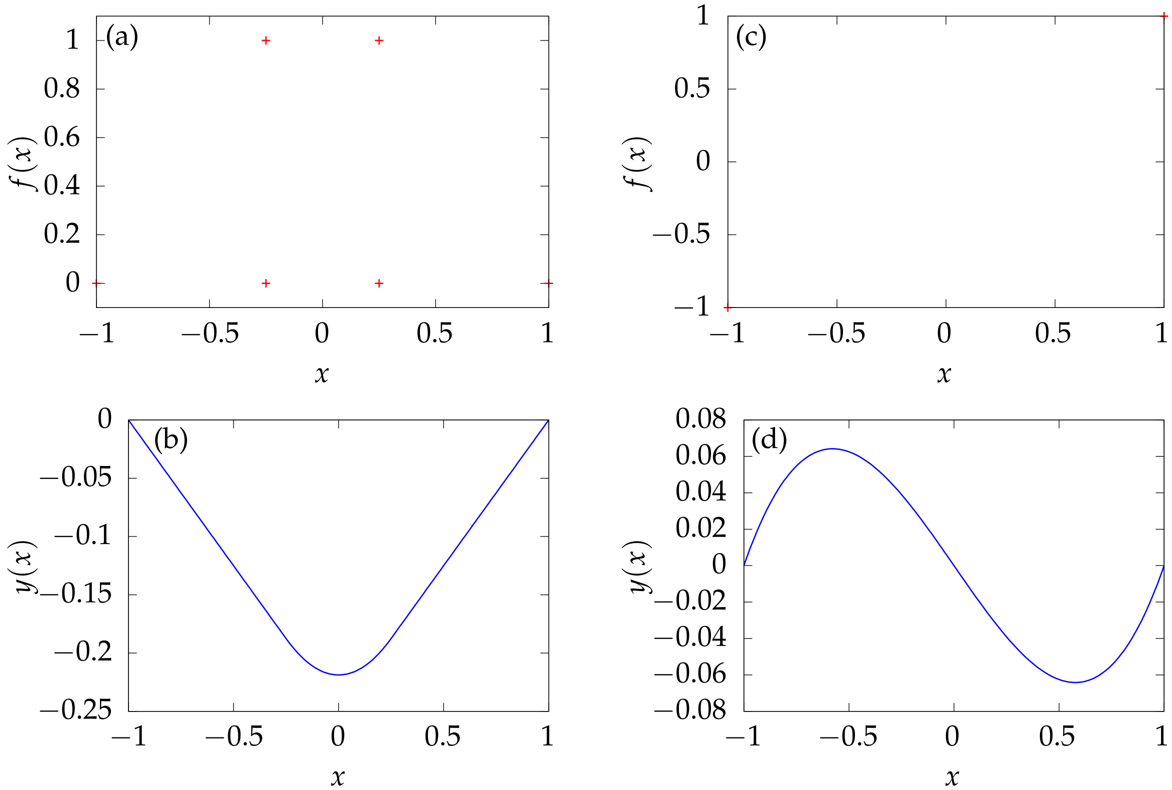

Figure 2: Solutions to the differential equation in question 1 for two different forcing functions $f(x)$.

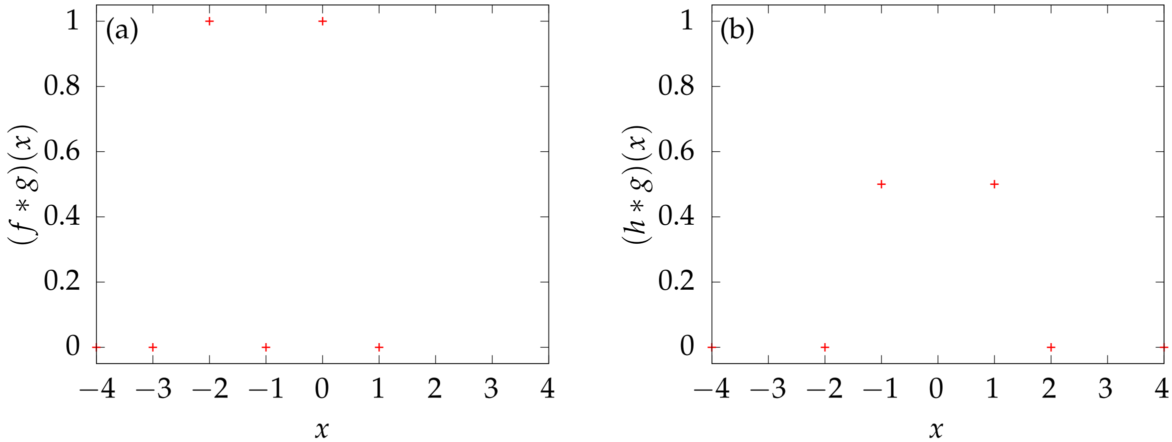

Figure 3: Convolutions of (a) $f * g$ and (b) $h * g$ considered in question 2.

(d) For $f(x) = x$, the solution is given by

$$\begin{aligned} y(x) &= \int_{-1}^{1} G(x,x')f(x')dx' \\ &= \int_{-1}^{x} G(x,x')f(x')dx' + \int_{x}^{1} G(x,x')f(x')dx' \\ &= \frac{x-1}{2}\int_{-1}^{x} x'(x'+1)dx' + \frac{x+1}{2}\int_{x}^{1} x'(x'-1)dx' \\ &= \frac{x-1}{2}\left[\frac{x'^3}{3} + \frac{x'^2}{2}\right]_{-1}^{x} + \frac{x+1}{2}\left[\frac{x'^3}{3} - \frac{x'^2}{2}\right]_{x}^{1} \\ &= \frac{x^3}{6} - \frac{x}{6}. \end{aligned}$$

The input function and solution are plotted in Figs. 2(c) and 2(d) respectively. It can be seen that

$$y'(x) = \frac{x^2}{2} - \frac{1}{6}, \qquad y''(x) = x$$

and thus the solution satisfies the differential equation. In addition,

$$y(\pm 1) = \frac{\pm 1}{6} - \frac{\pm 1}{6} = 0.$$

and thus the boundary conditions are satisfied.

2. The first convolution is given by

$$
\begin{aligned}
(f * g)(x) &= \int_{-\infty}^{\infty} f(\xi) g(x-\xi) d\xi \\
&= \int_{-\infty}^{\infty} (\delta(\xi-2) - \delta(\xi) - \delta(\xi+2)) g(x-\xi) d\xi \\
&= g(x-2) + g(x) + g(x+2)
\end{aligned}
$$

and hence the convolution consists of three copies of g, so

$$
g(x) = \begin{cases} 1-|x+2| & \text{for } -3 < x < -1, \\ 1-|x| & \text{for } -1 \leq x < 1, \\ 1-|x-2| & \text{for } 1 \leq x < 3, \\ 0 & \text{otherwise.} \end{cases}
$$

The function is plotted in Fig. 3(a). The second convolution is given by

$$
(h * g)(x) = \int_{-\infty}^{\infty} f(2\xi) g(x-\xi) d\xi.
$$

Making the substitution $2\xi = y$ gives

$$
\begin{aligned}
(h * g)(x) &= \int_{-\infty}^{\infty} f(y) g\left(x - \frac{y}{2}\right) \frac{dy}{2} \\
&= \frac{1}{2} \int_{-\infty}^{\infty} (\delta(y-2) + \delta(y) + \delta(y+2)) g\left(x - \frac{y}{2}\right) dy \\
&= \frac{g(x-1) + g(x) + g(x+1)}{2}.
\end{aligned}
$$

The solution consists of three copies of g, but now the peaks of g overlap. For the region $0 < x < 1$, note that

$$
\begin{aligned}
(h * g)(x) &= \frac{g(x-1) + g(x) + g(x+1)}{2} \\
&= \frac{0 + 1 - |x| + 1 - |x+1|}{2} \\
&= \frac{1 - (-x) + 1 - (x+1)}{2} \\
&= \frac{1}{2}.
\end{aligned}
$$

In addition, it can be seen that $(h * g)(-x) = (h * g)(x)$ and hence the function is even. An explicit form of the solution is therefore

$$
(h * g)(x) = \begin{cases} 1/2 & \text{for } |x| < 1, \\ 1 - x/2 & \text{for } 1 \leq |x| < 2, \\ 0 & \text{for } |x| \geq 2. \end{cases}
$$

The function is plotted in Fig. 3(b).

3. The first function is given by

$$f_1(x) = \int_{-\infty}^{\infty} f_0(\xi) f_0(x-\xi) d\xi = \int_{x-1}^{x} f_0(\xi) d\xi.$$

Evaluating this integral can be carried out by determining the size of the non-zero part of $f_0(x)$ in the range $x-1 < \xi < x$. If $0 < x < 1$ then

$$f_1(x) = \int_{x-1}^{0} 0 d\xi + \int_{0}^{x} 1 d\xi = 0 + [\xi]_0^x = x$$

and if $1 \leq x < 2$ then

$$f_1(x) = \int_{x-1}^{1} 1 d\xi + \int_{1}^{x} 0 d\xi = [\xi]_{x-1}^1 + 0 = 2 - x.$$

For other values of x, the intervals $x-1 < \xi < x$ and $0 < \xi < 1$ will not overlap and hance f_1 will be zero. Hence

$$f_1(x) = \begin{cases} x & \text{if } 0 < x < 1, \\ 2 - x & \text{if } 1 \leq x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

The second function is given by

$$f_2(x) = \int_{-\infty}^{\infty} f_1(\xi) f_0(x-\xi) d\xi = \int_{x-1}^{x} f_1(\xi) d\xi.$$

For $0 < x < 1$ this is

$$f_2(x) = \int_0^x \xi \, d\xi = \frac{x^2}{2}.$$

For $1 \leq x < 2$ this is

$$\begin{aligned} f_2(x) &= \int_{x-1}^{1} \xi \, d\xi + \int_{1}^{x} (2-\xi) d\xi \\ &= \left[\frac{\xi^2}{2}\right]_{x-1}^{1} + \left[2\xi - \frac{\xi^2}{2}\right]_1^x \\ &= \frac{1 - (x-1)^2 + 4x - x^2 - 4 + 1}{2} \\ &= \frac{-2x^2 + 6x - 3}{2}. \end{aligned}$$

For $2 \leq x < 3$ this is

$$f_2(x) = \int_{x-1}^{2} (2-\xi) d\xi = \frac{(3-x)^2}{2}$$

and hence the general solution is

$$2f_2(x) = \begin{cases} x^2 & \text{for } 0 < x < 1, \\ -2x^2 + 6x - 3 & \text{for } 1 \leq x < 2, \\ (3-x)^2 & \text{for } 2 \leq x < 3, \\ 0 & \text{otherwise.} \end{cases}$$

It can be verified that f_2 is symmetric about $x = 3/2$, and thus $f_2(3 - x) = f_2(x)$. The third function is given by

$$f_3(x) = \int_{-\infty}^{\infty} f_1(\xi) f_0(x - \xi) d\xi = \int_{x-1}^{x} f_1(\xi) d\xi.$$

For $0 < x < 1$ this is

$$f_3(x) = \int_0^x \frac{\xi^2}{2} d\xi = \frac{x^3}{6}.$$

For $1 \leq x < 2$ this is

$$\begin{aligned} f_3(x) &= \int_{x-1}^{1} \frac{\xi^2}{2} d\xi + \int_1^x \frac{-2\xi^2 + 6\xi - 3}{2} d\xi \\ &= \left[\frac{\xi^3}{6}\right]_{x-1}^{1} + \left[\frac{-2\xi^3 + 9\xi^2 - 9\xi}{6}\right]_1^x \\ &= \frac{1 - (x-1)^3}{6} + \frac{-2x^3 + 9x^2 - 9x + 2 + 9 - 9}{6} \\ &= \frac{1 - x^3 + 3x^2 - 3x + 1 - 2x^3 + 9x^2 - 9x + 2}{6} \\ &= \frac{-3x^3 + 12x^2 - 12x + 4}{6}. \end{aligned}$$

Now consider the case for $x \geq 2$: rather than integrating directly, the symmetry of f_2 can be exploited. Starting from the definition

$$f_3(x) = \int_{x-1}^{x} f_2(\xi) d\xi$$

and making use of the substitution $\xi = 3 - \gamma$ gives

$$f_3(x) = -\int_{4-x}^{3-x} f_2(3-\gamma) d\gamma = \int_{(4-x)-1}^{4-x} f_2(\gamma) d\gamma = f_3(4-x).$$

Hence for $2 \leq x < 3$ the solution is given by

$$\begin{aligned} f_3(x) &= f_3(4-x) \\ &= \frac{-3(4-x)^3 + 12(4-x)^2 - 12(4-x) + 4}{6} \\ &= \frac{-192 + 144x - 36x^2 + 3x^3 + 192 - 96x + 12x^2 - 48 + 12x + 4}{6} \\ &= \frac{3x^3 - 24x^2 + 60x - 44}{6} \end{aligned}$$

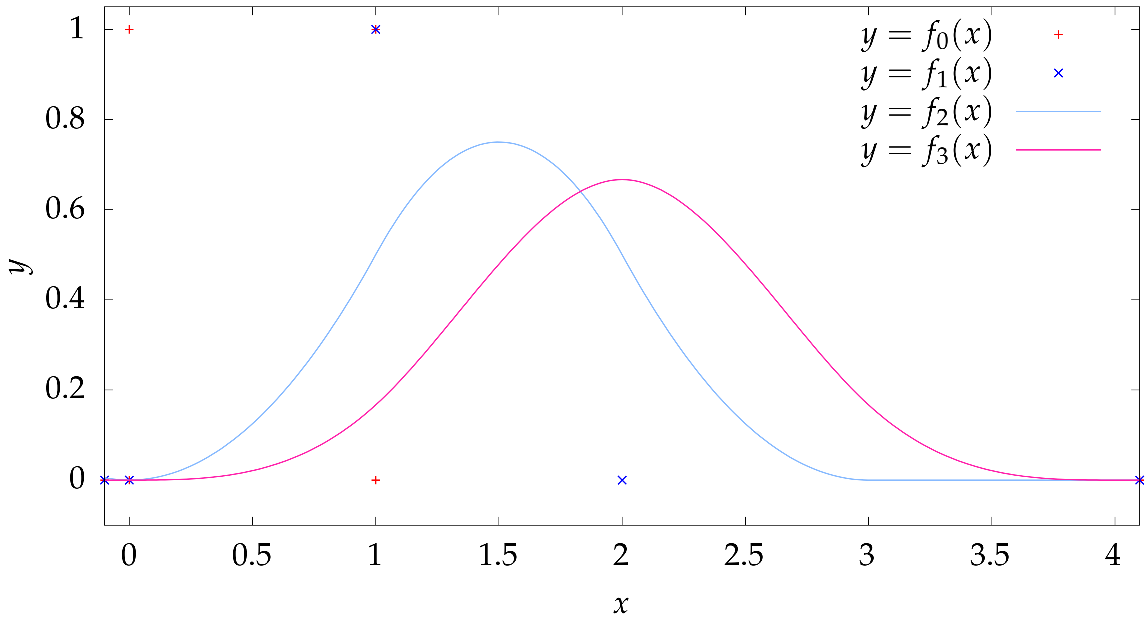

Figure 4: A sequence of convolutions considered in question 3.

and for $3 \leq x < 4$ the solution is given by

$$f_3(x) = f_3(4-x) = \frac{(4-x)^3}{6}.$$

Hence a complete solution is

$$6f_3(x) = \begin{cases} x^3 & \text{for } 0 \leq x < 1, \\ -3x^3 + 12x^2 - 12x + 4 & \text{for } 1 \leq x < 2, \\ 3x^3 - 24x^2 + 60x - 44 & \text{for } 2 \leq x < 3, \\ (4-x)^3 & \text{for } 3 \leq x < 4, \\ 0 & \text{otherwise.} \end{cases}$$

The functions are plotted in Fig. 4, it can be seen that as each successive convolution is applied, the functions are smoothed out and moved to the right. The functions begin to look like Gaussians. The area under each curve is equal to one.

4. The functions are shown in Fig. 5 for the cases of $a = 1$ and $a = 2$. It can be seen that they are odd, and hence it is immediately known that that the cosine terms in the

Fourier series are zero. The sine terms are given by

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx \\
&= \frac{2}{\pi}\int_{0}^{a} (a-x)\sin nx\,dx \\
&= \frac{2}{\pi}\int_{0}^{a} \frac{-\cos nx}{n}dx + \frac{2}{\pi}\left[\frac{-(a-x)\cos nx}{n}\right]_0^a \\
&= -\frac{2}{\pi}\left[\frac{\sin nx}{n^2}\right]_0^a + \frac{2a}{n\pi} \\
&= \frac{2(na-\sin na)}{n^2\pi}.
\end{aligned}$$

Hence

$$f(x) = \sum_{n=1}^{\infty} \frac{2(na-\sin na)\sin nx}{n^2\pi}.$$

Comparisons to the exact forms are shown in Fig. 5 for the cases $a = 1$ and $a = 2$.

5. For 14.2.22, $u = y$ and $v = x$. Since

$$\frac{\partial u}{\partial x} = 0, \qquad \frac{\partial v}{\partial y} = 0$$

it can be seen that the first Cauchy–Riemann equation is satisfied. However

$$\frac{\partial u}{\partial y} = 1, \qquad -\frac{\partial v}{\partial x} = -1.$$

and thus the second Cauchy–Riemann equation is not satisfied. This should be expected, since if $z = x + iy$ the function can be written as $i\bar{z}$ and thus it is not analytic.

For 14.2.23, the real and imaginary components are given by

$$u = \frac{x}{x^2+y^2}, \qquad v = \frac{-y}{x^2+y^2}.$$

The two sides of the first Cauchy–Riemann equation are

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} - \frac{2x^2}{x^2+y^2} = \frac{y^2-x^2}{x^2+y^2}, \qquad \frac{\partial v}{\partial y} = -\frac{1}{x^2+y^2} + \frac{2y^2}{x^2+y^2} = \frac{y^2-x^2}{x^2+y^2}$$

and thus it is satisfied. The two sides of the second equation are

$$\frac{\partial u}{\partial y} = -\frac{2xy}{x^2+y^2}, \qquad -\frac{\partial v}{\partial x} = -\frac{2xy}{x^2+y^2}$$

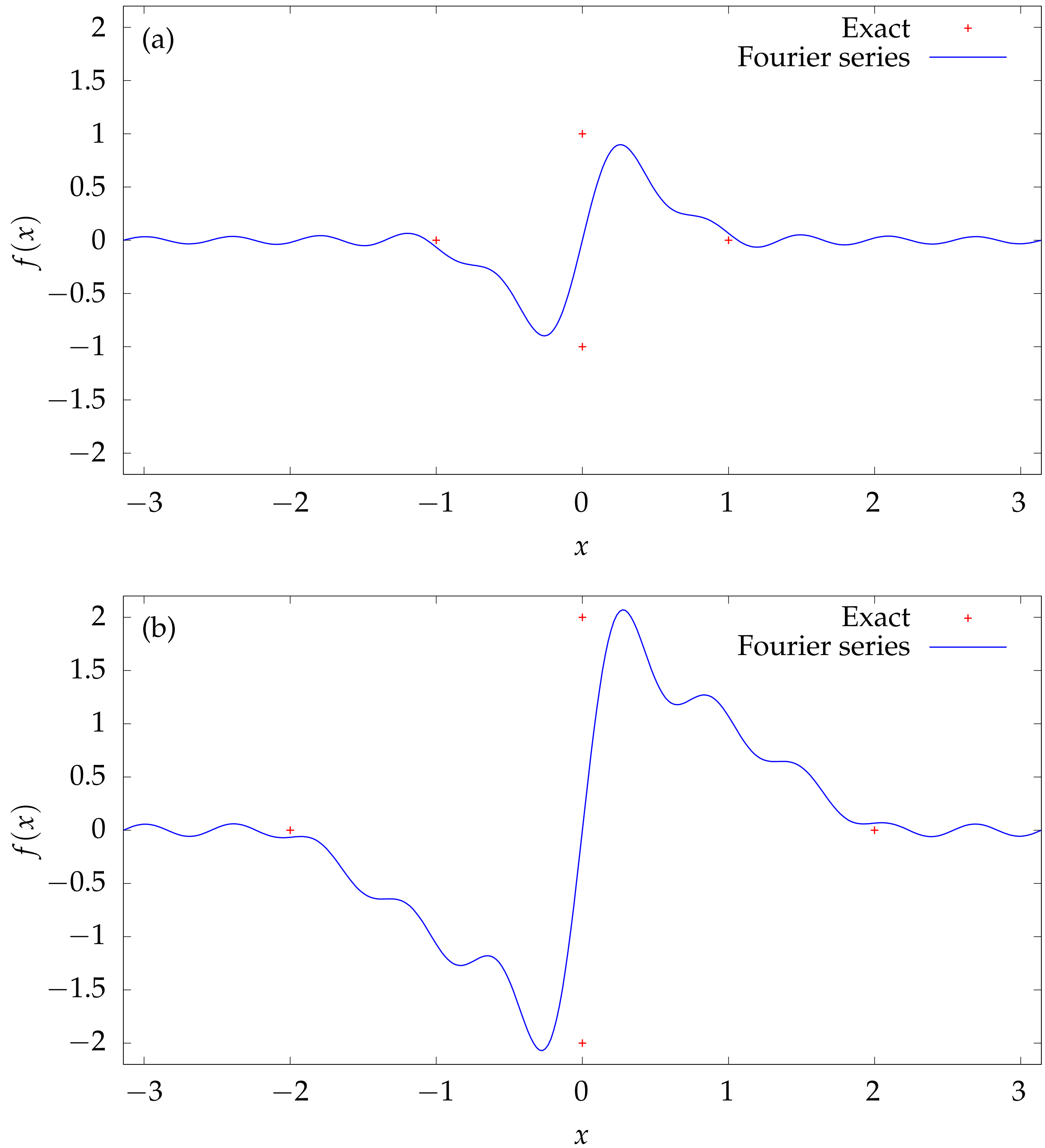

Figure 5: The function considered in question 4 and the first ten terms of its Fourier series, for the cases of (a) $a = 1$ and (b) $a = 2$.

and they are both equal. This should be expected since the function can be written as $1/z$, and is therefore analytic.

For 14.2.24, the real and imaginary components are given by

$$u = \frac{y}{x^2+y^2}, \qquad v = -\frac{x}{x^2+y^2}$$

The two sides of the first Cauchy–Riemann equation are

$$\frac{\partial u}{\partial x} = -\frac{2xy}{x^2+y^2}, \qquad \frac{\partial v}{\partial y} = \frac{2xy}{x^2+y^2}$$

and thus it is not satisfied, due to a sign difference. For the second equation

$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2} - \frac{2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}, \qquad -\frac{\partial v}{\partial x} = \frac{1}{x^2+y^2} - \frac{2x^2}{x^2+y^2} = \frac{y^2-x^2}{x^2+y^2}$$

and thus the equation is satisfied. Since the function can be written as $-i/\bar{z}$ it is not analytic, so it should be expected that at least one of the Cauchy–Riemann equations does not hold.

6. Polar coordinates (r, θ) can be linked to cartesian coordinates according to $x = r\cos\theta$, $y = r\sin\theta$. Hence, by using the chain rule,

$$\frac{\partial u}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial u}{\partial y} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y} \tag{1}$$

and

$$\frac{\partial u}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial u}{\partial y} = -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y} \tag{2}$$

Equations 1 and 2 can be viewed as simultaneous equations for $\partial u/\partial x$ and $\partial v/\partial y$. By using standard methods of solution, it can be seen that

$$\frac{\partial u}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} \tag{3}$$

and

$$\frac{\partial u}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} \tag{4}$$

Since the above derivation will apply for an arbitrary function, Eqs. 3 and 4 will also hold if u is replaced by v. Hence the first Cauchy–Riemann equation can be written as

$$\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} = \sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta} \tag{5}$$

and the second Cauchy–Riemann equation can be written as

$$\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} = -\cos\theta\frac{\partial v}{\partial r} + \frac{\sin\theta}{r}\frac{\partial v}{\partial \theta}. \tag{6}$$

These equations can be expressed in a simpler form. Taking $\cos\theta$ times Eq. 5 plus $\sin\theta$ times Eq. 6 gives

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}.$$

Taking $-\sin\theta$ times Eq. 5 plus $\cos\theta$ times Eq. 6 gives

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

7. By making use of the expression for the Laplacian in polar coordinates it can be seen that

$$\begin{aligned}
\nabla^2 u &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial \theta}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(r\frac{\partial u}{\partial r}\right) \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial \theta}\right) - \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\right) \\
&= 0
\end{aligned}$$

where the final line has been obtained by assuming that the second-order partial derivatives commute. Similarly

$$\begin{aligned}
\nabla^2 v &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} \\
&= -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(r\frac{\partial v}{\partial r}\right) \\
&= -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial \theta}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial r}\right) \\
&= 0
\end{aligned}$$

and thus both components satisfy the Laplace equation in polar coordinates.